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Preface

The motivation for these lecture notes on minimal surfaces is to cover the necessary background material needed for the papers [CM3], [CM8], [CM9], and [CM6] on compactness and convergence of minimal surfaces in three-manifolds. Some of these results are described in the last chapter of these notes. These results about convergence and compactness of embedded minimal surfaces in three-manifolds are in part motivated by a question of Pitts and Rubinstein. This question asks to give a bound for the Morse index of all embedded closed minimal surfaces of fixed genus in a closed three-manifold; see Chapter 5 for the precise statement. The claim of Pitts and Rubinstein is that if there is such a bound for a sufficiently large class of metrics on $\mathbf{S}^3$, then the famous spherical space-form problem can be settled affirmatively.

We also hope that these notes will help to stimulate interaction between minimal surface theory and the topology of three-manifolds.

These notes are an expanded version of a one-semester course taught at Courant in the spring of 1998. The only prerequisites needed are a basic knowledge of Riemannian geometry and some familiarity with the maximum principle. Of the various ways of approaching minimal surfaces (from complex analysis, PDE, or geometric measure theory), we have chosen to focus on the PDE aspects of the theory.

In Chapter 1, we will first derive the minimal surface equation as the Euler-Lagrange equation for the area functional on graphs. Subsequently, we derive the parametric form of the minimal surface equation (the first variation formula). The focus of the first chapter is on the basic properties of minimal surfaces, including the monotonicity formula for area and the Bernstein theorem. We also mention some examples. In the last section of Chapter 1, we derive the second variation formula, the stability inequality, and define the Morse index of a minimal surface.

Chapter 2 deals with generalizations of the Bernstein theorem discussed in Chapter 1. We begin the chapter by deriving Simons' inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbb{R}^n$. In the later sections, we discuss various applications of such an inequality. The first application that we give is to a theorem of Choi-Schoen giving curvature estimates for minimal surfaces with small total curvature. Using this estimate, we give a short proof of Heinz's curvature estimate for minimal graphs. Next, we discuss a priori estimates for stable minimal surfaces in three-manifolds, including estimates on area and total curvature of Colding-Minicozzi and the curvature estimate of Schoen. After that, we follow Schoen-Simon-Yau and combine Simons' inequality with the stability inequality to show higher L^p bounds for the square of the norm of the second fundamental form for stable minimal hypersurfaces. The higher L^p bounds are then used together with Simons' inequality to show curvature estimates for stable minimal hypersurfaces and to give a generalization due to De Giorgi, Almgren, and Simons of the Bernstein theorem proven in Chapter 1. We close the chapter with a discussion of minimal cones in Euclidean space and the relationship to the Bernstein theorem.

We start Chapter 3 by introducing stationary varifolds as a generalization of classical minimal surfaces. After that, we prove a generalization of the Bernstein theorem for minimal surfaces discussed in the preceding chapter. Namely, following [CM4], we will show in Chapter 3 that, in fact, a bound on the density gives an upper bound for the smallest affine subspace that the minimal surface lies in. We will deduce this theorem from the properties of the coordinate functions (in fact, more generally properties of harmonic functions) on k-rectifiable stationary varifolds of arbitrary codimension in Euclidean space.

Chapter 4 discusses the solution to the classical Plateau problem, focusing primarily on its regularity. The first three sections cover the basic existence result for minimal disks. After some general discussion of unique continuation and nodal sets, we study the local description of minimal surfaces in a neighborhood of either a branch point or a point of nontransverse intersection. Following Osserman and Gulliver, we rule out interior branch points for solutions of the Plateau problem. In the remainder of the chapter, we prove the embeddedness of the solution to the Plateau problem when the boundary is in the boundary of a mean convex domain. This last result is due to Meeks and Yau.

Finally, in Chapter 5, we discuss the theory of minimal surfaces in three-manifolds. We begin by explaining how to extend the earlier results to this case (in particular, monotonicity, the strong maximum principle, and some of the other basic estimates for minimal surfaces). Next, we prove the compactness theorem of Choi and Schoen for embedded minimal surfaces in three-manifolds with positive Ricci curvature. An important point for this compactness result is that by results of Choi-Wang and Yang-Yau such minimal surfaces have uniform area bounds. The next section surveys recent results of [CM3], [CM8], [CM9], and [CM6] on compactness and convergence of minimal surfaces without area bounds. Finally, in the last section, we mention an application (from [CM7]) of the ideas of [CM8] to the study of complete minimal surfaces in $\mathbb{R}^3$.

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The Variation Formulas and Some Consequences

In this chapter, we will first derive the minimal surface equation as the Euler-Lagrange equation for the area functional on graphs. Subsequently, we derive the parametric form of the minimal surface equation (the first variation formula). The focus of the chapter is on some basic properties of minimal surfaces, including the monotonicity formula for area and the Bernstein theorem. We also mention some examples. In the last section, we derive the second variation formula, the stability inequality, and define the Morse index of a minimal surface.

1.1 The Minimal Surface Equation and Minimal Submanifolds

Suppose that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and consider the graph of the function u

$$\text{Graph}_u = \{(x, y, u(x, y)) \mid (x, y) \in \Omega\} . \tag{1.1}$$

Then the area is

$$\begin{aligned} \text{Area}(\text{Graph}_u) &= \int_\Omega |(1, 0, u_x) \times (0, 1, u_y)| \\ &= \int_\Omega \sqrt{1 + u_x^2 + u_y^2} = \int_\Omega \sqrt{1 + |\nabla u|^2} \, , \end{aligned} \tag{1.2}$$

and the (upward pointing) unit normal is

$$N = \frac{(1, 0, u_x) \times (0, 1, u_y)}{|(1, 0, u_x) \times (0, 1, u_y)|} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}} \, . \tag{1.3}$$

Therefore for the graphs $\text{Graph}_{u+t\eta}$ where $\eta|\partial\Omega = 0$ we get that

$$\text{Area}(\text{Graph}_{u+t\eta}) = \int_\Omega \sqrt{1 + |\nabla u + t\,\nabla\eta|^2} \tag{1.4}$$

hence

$$\frac{d}{dt}_{t=0} \text{Area}(\text{Graph}_{u+t\eta}) = \int_\Omega \frac{\langle \nabla u \, , \nabla \eta \rangle}{\sqrt{1+|\nabla u|^2}} = - \int_\Omega \eta \text{ div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) . \tag{1.5}$$

Therefore the graph of u is a critical point for the area functional if u satisfies the divergence form equation

$$\text{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0 \, . \tag{1.6}$$

Equation (1.6) is the divergence form of the *minimal surface equation* and can alternatively be written as

$$0 = \left(1+|\nabla u|^2\right)^{\frac{3}{2}} \left[\left(\frac{u_x}{\sqrt{1+|\nabla u|^2}} \right)_x + \left(\frac{u_y}{\sqrt{1+|\nabla u|^2}} \right)_y \right] \tag{1.7}$$

$$= (1+u_y^2)\, u_{xx} + (1+u_x^2)\, u_{yy} - 2\, u_x\, u_y\, u_{xy} \, .$$

Next we want to show that the graph of a function on Ω satisfying the minimal surface equation is not just a critical point for the area functional but is actually area-minimizing amongst surfaces in the cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^3$. Let ω be the two-form on $\Omega \times \mathbb{R}$ given by that for $X,\, Y \in \mathbb{R}^3$

$$\omega(X,Y) = \text{det}(X,Y,N) \, , \tag{1.8}$$

where

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1+|\nabla u|^2}} \, . \tag{1.9}$$

Observe that

$$\omega \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \frac{1}{\sqrt{1+|\nabla u|^2}} \begin{vmatrix} 1 & 0 & -u_x \\ 0 & 1 & -u_y \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{1+|\nabla u|^2}} \, , \tag{1.10}$$

$$\omega \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \frac{1}{\sqrt{1+|\nabla u|^2}} \begin{vmatrix} 0 & 0 & -u_x \\ 1 & 0 & -u_y \\ 0 & 1 & 1 \end{vmatrix} = \frac{-u_x}{\sqrt{1+|\nabla u|^2}} \, , \tag{1.11}$$

and

$$\omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = \frac{1}{\sqrt{1+|\nabla u|^2}} \begin{vmatrix} 1 & 0 & -u_x \\ 0 & 0 & -u_y \\ 0 & 1 & 1 \end{vmatrix} = \frac{u_y}{\sqrt{1+|\nabla u|^2}} \,. \tag{1.12}$$

Hence

$$\omega = \frac{dx \wedge dy - u_x\, dy \wedge dz - u_y\, dz \wedge dx}{\sqrt{1+|\nabla u|^2}} \tag{1.13}$$

and

$$d\omega = \frac{\partial}{\partial x}\left(\frac{-u_x}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{-u_y}{\sqrt{1+|\nabla u|^2}}\right) = 0\,, \tag{1.14}$$

since u satisfies the minimal surface equation. In sum, the form ω is closed and, given any orthogonal unit vectors X and Y at a point (x, y, z),

$$|\omega(X, Y)| \leq 1\,, \tag{1.15}$$

where equality holds if and only if

$$X, Y \subset T_{(x,y,u(x,y))} \operatorname{Graph}_u \,. \tag{1.16}$$

Such a form ω is called a *calibration*. From this, we have that if $\Sigma \subset \Omega \times \mathbb{R}$ is any other surface with $\partial\Sigma = \partial \operatorname{Graph}_u$, then by Stokes' theorem since ω is closed,

$$\operatorname{Area}(\operatorname{Graph}_u) = \int_{\operatorname{Graph}_u} \omega = \int_\Sigma \omega \leq \operatorname{Area}(\Sigma)\,. \tag{1.17}$$

This shows that Graph_u is area-minimizing amongst such surfaces. If $\Omega \subset \mathbb{R}^2$ contains a ball of radius r, then, since $\partial B_r \cap \operatorname{Graph}_u$ divides ∂B_r into two components at least one of which has area at most equal to $(\operatorname{Area}(\mathbf{S}^2)/2)\, r^2$, we get from (1.17) the crude estimate

$$\operatorname{Area}(B_r \cap \operatorname{Graph}_u) \leq \frac{\operatorname{Area}(\mathbf{S}^2)}{2}\, r^2\,. \tag{1.18}$$

If the domain Ω is convex, the minimal graph is absolutely area-minimizing. To see this, observe first that for a convex set Ω the nearest point projection $P : \mathbb{R}^3 \to \Omega \times \mathbb{R}$ is a distance nonincreasing Lipschitz map that is equal to the identity on $\Omega \times \mathbb{R}$. If $\Sigma \subset \mathbb{R}^3$ is any other surface with $\partial\Sigma = \partial \operatorname{Graph}_u$, then $\Sigma' = P(\Sigma)$ has $\operatorname{Area}(\Sigma') \leq \operatorname{Area}(\Sigma)$. Applying (1.17) to Σ', we see that $\operatorname{Area}(\operatorname{Graph}_u) \leq \operatorname{Area}(\Sigma')$ and the claim follows.

Very similar calculations to the ones above show that if $\Omega \subset \mathbb{R}^{n-1}$ and $u : \Omega \to \mathbb{R}$ is a C^2 function, then the graph of u is a critical point for the area functional if and only if u satisfies the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0\,. \tag{1.19}$$

Moreover, as in (1.17), if u satisfies (1.19), then the graph of u is actually area-minimizing. Consequently, as in (1.18), if Ω contains a ball of radius r, then

$$\operatorname{Vol}(B_r \cap \operatorname{Graph}_u) \le \frac{\operatorname{Vol}(\mathbf{S}^{n-1})}{2}\, r^{n-1}\,. \tag{1.20}$$

We could also have looked more generally for a k-dimensional submanifold Σ possibly with boundary and sitting inside some Riemannian manifold M and which is a critical point for the area functional. In the following, if X is a vector field on $\Sigma \subset M$, then we let X^T and X^N denote the tangential and normal components, respectively. A will denote the second fundamental form of Σ. That is, A is the vector-valued symmetric bilinear form on Σ such that for X, $Y \in T_x\Sigma$,

$$A(X,Y) = (\nabla_X Y)^N\,. \tag{1.21}$$

Observe that

$$\sum_{\ell=1}^{n-k} g(A(X,Y), N_\ell)\, N_\ell = \sum_{\ell=1}^{n-k} g(\nabla_X Y, N_\ell)\, N_\ell = -\sum_{\ell=1}^{n-k} g(Y, \nabla_X N_\ell)\, N_\ell\,, \tag{1.22}$$

where N_ℓ is an orthonormal basis of vector fields for the normal space to Σ in a neighborhood of x.

The *mean curvature vector* H at x is by definition

$$H = \sum_{i=1}^{k} A(E_i, E_i)\,, \tag{1.23}$$

where E_i is an orthonormal basis for $T_x\Sigma$. Furthermore, the *norm squared of the second fundamental form* at x is by definition

$$|A|^2 = \sum_{i,j=1}^{k} |A(E_i, E_j)|^2\,. \tag{1.24}$$

Recall also that the Gauss equations assert that if X, $Y \in T_x\Sigma$, then

$$\begin{aligned} &K_\Sigma(X,Y)\,|X\wedge Y|^2 \\ &\quad = K_M(X,Y)\,|X\wedge Y|^2 + g(A(X,X),\, A(Y,Y)) - g(A(X,Y),\, A(X,Y))\,, \end{aligned} \tag{1.25}$$

where $|X \wedge Y|^2 = g(X,X)\,g(Y,Y) - g(X,Y)^2$ and $K_M(X,Y)$ and $K_\Sigma(X,Y)$ are the sectional curvatures of M and Σ, respectively, in the two-plane spanned by the vectors X and Y. If $\Sigma^{n-1} \subset M^n$ is a hypersurface and N is a unit normal vector field in a neighborhood of x, then

$$\nabla_{(\cdot)} N : T_x\Sigma \to T_x\Sigma \tag{1.26}$$

is a symmetric map (often referred to as the Weingarten map) and its eigenvalues $(\kappa_i)_{i=1,\dots,n-1}$ are called the principal curvatures. Moreover,

$$g(H,N) = -\sum_{i=1}^{n-1} \kappa_i\,. \tag{1.27}$$

Finally, if X is a vector field defined in a neighborhood of Σ, then the *divergence* of X at $x \in \Sigma$ is

$$\mathrm{div}_\Sigma\, X = \sum_{i=1}^{n-1} g(\nabla_{E_i} X, E_i)\,, \tag{1.28}$$

where E_i is an orthonormal basis for $T_x\Sigma$.

Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a variation of Σ with compact support and fixed boundary. That is, $F = \mathrm{Id}$ outside a compact set,

$$F(x,0) = x\,, \tag{1.29}$$

and for all $x \in \partial\Sigma$

$$F(x,t) = x\,. \tag{1.30}$$

The vector field F_t restricted to Σ is often called the *variational vector field*. Now we want to compute the first variation of area for this one-parameter family of surfaces. Let x_i be local coordinates on Σ. Set $g_{i,j}(t) = g(F_{x_i}, F_{x_j})$ and $\nu_t = \sqrt{\det(g_{i,j}(t))}\,\sqrt{\det(g^{i,j}(0))}$, where $a^{i,j}$ denotes the inverse of the matrix $a_{i,j}$. Note that ν_t is well-defined independent of the choice of coordinate system. Furthermore,

$$\mathrm{Vol}(F(\Sigma,t)) = \int \nu_t \sqrt{\det(g_{i,j}(0))}\,, \tag{1.31}$$

and therefore

$$\frac{d}{dt}_{t=0} \mathrm{Vol}(F(\Sigma,t)) = \int \frac{d}{dt}_{t=0} \nu_t \sqrt{\det(g_{i,j}(0))}\,. \tag{1.32}$$

To evaluate $d/dt_{t=0}\nu_t$ at some point x, we may choose the coordinate system such that at x it is orthonormal. Using this and the fact that $\nabla_{F_t} F_{x_i} - \nabla_{F_{x_i}} F_t =$

$[F_t, F_{x_i}] = 0$, we get at x,

$$\begin{aligned}
\frac{d}{dt}_{t=0} \nu_t &= \frac{1}{2} \sum_{i=1}^{k} \frac{d}{dt} g(F_{x_i}, F_{x_i}) = \sum_{i=1}^{k} g(\nabla_{F_t} F_{x_i}, F_{x_i}) \\
&= \sum_{i=1}^{k} g(\nabla_{F_{x_i}} F_t, F_{x_i}) \\
&= \sum_{\ell=1}^{n-k} \sum_{i=1}^{k} g(\nabla_{F_i} g(F_t, N_\ell)\, N_\ell, F_{x_i}) + \operatorname{div}_\Sigma\ F_t^T \\
&= \sum_{\ell=1}^{n-k} \sum_{i=1}^{k} g(F_t, N_\ell)\, g(\nabla_{F_{x_i}} N_\ell, F_{x_i}) + \operatorname{div}_\Sigma\ F_t^T \\
&= g(F_t, H) + \operatorname{div}_\Sigma\ F_t^T\,.
\end{aligned} \tag{1.33}$$

Here N_ℓ is an orthonormal basis for the normal bundle of Σ at x. Integrating gives by Stokes' theorem

$$\frac{d}{dt}_{t=0} \operatorname{Vol}(F(\Sigma, t)) = \int_\Sigma g(F_t, H)\,. \tag{1.34}$$

Hence Σ is a critical point for the area functional if and only if the mean curvature H vanishes identically.

Definition 1.1 (Minimal Submanifold) An immersed submanifold $\Sigma^k \subset M^n$ is said to be *minimal* if the mean curvature H vanishes identically.

It follows from the identity (1.34) that a graph in $\mathbb{R}^3$ is a minimal surface if and only if it satisfies the minimal surface equation (1.6).

1.2 Some Simple Examples of Minimal Surfaces in $\mathbb{R}^3$

Example 1.2 (A Plane) $z = 0$.

Example 1.3 (The Helicoid) $z = \tan^{-1}\left(\frac{y}{x}\right)$ which is given in parametric form by $(x, y, z) = (t\ \cos s, t\ \sin s, s)$ where s, $t \in \mathbb{R}$.

Example 1.4 (The Catenoid) $z = \cosh^{-1}\sqrt{x^2 + y^2}$, that is, the surface obtained by rotating the curve $x = \cosh z$ around the z-axis.

Example 1.5 (Scherk's Surface) Scherk's surface is the union of the closure of the surfaces

$$\Sigma_{k,\ell} = \left\{ (x, y, z) \mid |x - k| < 1, |y - \ell| < 1,\ \text{and}\ z = \log \frac{\cos\frac{\pi}{2}(y - \ell)}{\cos\frac{\pi}{2}(x - k)} \right\}, \tag{1.35}$$

where k, ℓ are even and $k + \ell \equiv 0 \mod 4$.

Let us check that Scherk's surface is, in fact, a minimal surface. Since each surface $\Sigma_{k,\ell}$ is given as a graph of $z = z(x, y)$, we need only check that $z = z(x, y)$ satisfies the minimal surface equation. Clearly $z_x = \frac{\pi}{2} \tan \frac{\pi}{2} (x - k)$, $z_y = -\frac{\pi}{2} \tan \frac{\pi}{2} (y - \ell)$, $z_{xx} = \pi^2/4 + (\pi^2/4) \tan^2 \frac{\pi}{2} (x - k)$, $z_{yy} = -\pi^2/4 - (\pi^2/4) \tan^2 \frac{\pi}{2} (y - \ell)$, and $z_{xy} = 0$. Hence, z satisfies the minimal surface equation (1.7).

1.3 Consequences of the First Variation Formula

In this section, we will collect some important consequences of the first variation formula. The most important of these is the monotonicity formula, Proposition 1.8. In later chapters, we will return to this subject. In Chapter 3, we extend these results to stationary varifolds, and in Chapter 5 to minimal surfaces in a three-manifold.

Let X be a vector field on M, then

$$\text{div}_\Sigma X = \text{div}_\Sigma X^T + \sum_{\ell=1}^{n-k} g(X, N_\ell)\, g(H, N_\ell)\,. \tag{1.36}$$

From this and Stokes' theorem, we see that Σ is minimal if and only if for all vector fields X with compact support and vanishing on the boundary of Σ,

$$\int_\Sigma \text{div}_\Sigma X = 0\,. \tag{1.37}$$

Moreover, it follows directly from (1.36) that

$$\text{div}_\Sigma X = \text{div}_\Sigma X^T \tag{1.38}$$

for all vector fields X on Σ if and only if Σ is minimal.

In the following, if $x_0 \in \mathbb{R}^n$ is fixed, then we let $B_s = B_s(x_0)$ be the Euclidean ball of radius s centered at x_0.

As a consequence of (1.37), we will show the following proposition:

Proposition 1.6 (Harmonicity of the Coordinate Functions) $\Sigma^k \subset \mathbb{R}^n$ *is minimal if and only if the restrictions of the coordinate functions of* $\mathbb{R}^n$ *to* Σ *are harmonic functions.*

PROOF: Let η be a smooth function on Σ with compact support and $\eta|\partial\Sigma = 0$, then

$$\int_\Sigma \langle \nabla_\Sigma \eta, \nabla_\Sigma x_i \rangle = \int_\Sigma \langle \nabla_\Sigma \eta\,, e_i \rangle = \int_\Sigma \text{div}_\Sigma(\eta\, e_i)\,. \tag{1.39}$$

From this, the claim follows easily. ∎

Recall that if $\Xi \subset \mathbb{R}^n$ is a compact subset, then the smallest convex set containing Ξ (the convex hull, $\text{Conv}(\Xi)$) is the intersection of all half-spaces containing Ξ.

Proposition 1.7 (The Convex Hull Property) *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a compact minimal submanifold, then $\Sigma \subset \text{Conv}(\partial\Sigma)$.*

PROOF: As above, if e is a fixed vector of $\mathbb{R}^n$, then the function $u(x) = \langle e, x\rangle$ is a harmonic function on Σ. The claim now easily follows from the maximum principle. ∎

Before we state and prove the monotonicity formula of volume for minimal submanifolds, we will need to recall the coarea formula. This formula asserts (see, for instance, [Fe] for a proof) that if Σ is a manifold and $h : \Sigma \to \mathbb{R}$ is a proper (i.e., $h^{-1}((-\infty, t])$ is compact for all $t \in \mathbb{R}$) Lipschitz function on Σ, then for all locally integrable functions f on Σ and $t \in \mathbb{R}$

$$\int_{\{h\leq t\}} f\,|\nabla h| = \int_{-\infty}^{t} \int_{h=\tau} f\,d\tau\,. \tag{1.40}$$

Proposition 1.8 (The Monotonicity Formula) *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \mathbb{R}^n$; then for all $0 < s < t$*

$$t^{-k}\ \text{Vol}(B_t \cap \Sigma) - s^{-k}\ \text{Vol}(B_s \cap \Sigma) = \int_{(B_t \setminus B_s)\cap\Sigma} \frac{|(x-x_0)^N|^2}{|x-x_0|^{k+2}}\,. \tag{1.41}$$

PROOF: Since Σ is minimal,

$$\Delta_\Sigma |x-x_0|^2 = 2\ \text{div}_\Sigma(x-x_0) = 2k\,. \tag{1.42}$$

By Stokes' theorem integrating this gives

$$2\,k\ \text{Vol}(B_s \cap \Sigma) = \int_{B_s\cap\Sigma} \Delta_\Sigma |x-x_0|^2 = 2\int_{\partial B_s\cap\Sigma} |(x-x_0)^T|\,. \tag{1.43}$$

Using this and the coarea formula (i.e., (1.40)), an easy calculation gives

$$\begin{aligned} \frac{d}{ds}\left(s^{-k}\ \text{Vol}(B_s\cap\Sigma)\right) &= -k\,s^{-k-1}\ \text{Vol}(B_s\cap\Sigma) + s^{-k}\int_{\partial B_s\cap\Sigma} \frac{|x-x_0|}{|(x-x_0)^T|} \\ &= s^{-k-1}\int_{\partial B_s\cap\Sigma}\left(\frac{|x-x_0|^2}{|(x-x_0)^T|} - |(x-x_0)^T|\right) \\ &= s^{-k-1}\int_{\partial B_s\cap\Sigma} \frac{|(x-x_0)^N|^2}{|(x-x_0)^T|}\,. \end{aligned} \tag{1.44}$$

Integrating and applying the coarea formula once more gives the claim. ∎

As a corollary, we get the following:

Corollary 1.9 *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \mathbb{R}^n$; then the function*

$$\Theta_{x_0}(s) = \frac{\text{Vol}(B_s\cap\Sigma)}{\text{Vol}(B_1\subset\mathbb{R}^k)\,s^k} \tag{1.45}$$

is a nondecreasing function of s.

Moreover, if Σ is proper and $x_0 \in \Sigma$, then $\Theta_{x_0}(s) \geq 1$; if for some $s > 0$, $\Theta_{x_0}(s) = 1$, then $B_s \cap \Sigma$ is a ball in some k-dimensional plane.

PROOF: Proposition 1.8 directly shows that $\Theta_{x_0}(s)$ is monotone nondecreasing. Since Σ is smooth and proper, it is infinitesimally Euclidean and hence

$$\lim_{s \to 0} \Theta_{x_0}(s) \geq 1\,. \tag{1.46}$$

Combining monotonicity of $\Theta_{x_0}(s)$ with (1.46) shows that $\Theta_{x_0}(s) \geq 1$. If we have $\Theta_{x_0}(s) = 1$, then Θ_{x_0} is constant in s so that, by (1.41), $(x - x_0)^N$ is identically zero. Clearly this implies that Σ is dilation invariant, and since Σ is smooth, Σ is contained in a k-plane. ∎

For later reference, we will record some consequences of Corollary 1.9. Let Σ be a minimal submanifold and define the *density* at x_0 by

$$\Theta_{x_0} = \lim_{s \to 0} \Theta_{x_0}(s)\,. \tag{1.47}$$

This limit, which exists since $\Theta_{x_0}(s)$ is monotone, is always at least 1 for $x_0 \in \Sigma$ by (1.46). In fact, so long as Σ is smooth, Θ_{x_0} is a nonnegative integer equal to the multiplicity of Σ at x_0. Note that if Σ is not embedded, then this multiplicity can be greater than one.

The next result, which is an elementary consequence of monotonicity, shows that this multiplicity is upper semicontinuous.

Corollary 1.10 *If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, then the density Θ_x is an upper semicontinuous function on $\mathbb{R}^n$. Consequently, for any $\Lambda \geq 0$, the set*

$$\{x \in \Sigma \mid \Theta_x \geq \Lambda\} \tag{1.48}$$

is closed.

PROOF: We need to show that if x_j is a sequence of points going to x, then

$$\Theta_x \geq \limsup_{x_j \to x} \Theta_{x_j}\,. \tag{1.49}$$

Given any $\delta > 0$, there exists an $s > 0$ such that

$$\Theta_x \geq \Theta_x(2s) - \delta\,, \tag{1.50}$$

and we can choose $0 < \epsilon < s$ so that

$$\Theta_x \geq (1 + s^{-1}\,\epsilon)^k\,\Theta_x(2s) - 2\,\delta\,. \tag{1.51}$$

For any x_j with $|x - x_j| < \epsilon$,

$$\begin{aligned}\Theta_{x_j} \leq \Theta_{x_j}(s) \leq \frac{\mathrm{Vol}(B_{s+\epsilon}(x) \cap \Sigma)}{\mathrm{Vol}(B_1 \subset \mathbb{R}^k)\, s^k} &= (1 + s^{-1}\,\epsilon)^k\,\Theta_x(s+\epsilon) \\ \leq 2\delta + \Theta_x\,, &\end{aligned} \tag{1.52}$$

where the last inequality follows from (1.51). Since δ was arbitrarily small, (1.52) implies (1.49) and hence Θ is upper semicontinuous. It follows immediately that the set defined in (1.48) must be closed. ∎

Proposition 1.11 (The Mean Value Inequality) *If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, $x_0 \in \mathbb{R}^n$, and f is a function on Σ, then*

$$t^{-k} \int_{B_t \cap \Sigma} f - s^{-k} \int_{B_s \cap \Sigma} f \tag{1.53}$$

$$= \int_{(B_t \setminus B_s) \cap \Sigma} f \, \frac{|(x-x_0)^N|^2}{|x-x_0|^{k+2}} + \frac{1}{2} \int_s^t \tau^{-k-1} \int_{B_\tau \cap \Sigma} (\tau^2 - |x-x_0|^2) \, \Delta_\Sigma f \, d\tau \, .$$

PROOF: Observe that the monotonicity formula will be the special case where $f = 1$. Since Σ is minimal, integration by parts gives

(1.54)

$$2k \int_{B_s \cap \Sigma} f = \int_{B_s \cap \Sigma} f \, \Delta_\Sigma |x-x_0|^2$$

$$= \int_{B_s \cap \Sigma} |x-x_0|^2 \, \Delta_\Sigma f + 2 \int_{\partial B_s \cap \Sigma} f \, |(x-x_0)^T| - s^2 \int_{B_s \cap \Sigma} \Delta_\Sigma f \, .$$

Using this and the coarea formula (i.e., (1.40)) gives

(1.55)

$$\frac{d}{ds} \left(s^{-k} \int_{B_s \cap \Sigma} f \right)$$

$$= -k \, s^{-k-1} \int_{B_s \cap \Sigma} f + s^{-k} \int_{\partial B_s \cap \Sigma} f \, \frac{|x-x_0|}{|(x-x_0)^T|}$$

$$= s^{-k-1} \int_{\partial B_s \cap \Sigma} f \, \frac{|(x-x_0)^N|^2}{|(x-x_0)^T|} + \frac{1}{2} \, s^{-k-1} \int_{B_s \cap \Sigma} (s^2 - |x-x_0|^2) \, \Delta_\Sigma f \, .$$

Integrating and using the coarea formula gives the claim. ∎

For future reference, we next record a general mean value inequality which follows from Proposition 1.11.

Corollary 1.12 *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, $x_0 \in \Sigma$, and $s > 0$ satisfy $B_s(x_0) \cap \partial \Sigma = \emptyset$. If f is a nonnegative function on Σ with $\Delta_\Sigma f \geq -\lambda \, s^{-2} \, f$, then*

$$f(x_0) \leq \mathrm{e}^{\frac{\lambda}{2}} \, \frac{\int_{B_s \cap \Sigma} f}{\mathrm{Vol}(B_1 \subset \mathbb{R}^k) \, s^k} \, . \tag{1.56}$$

PROOF: If we define $g(t)$ by

$$g(t) = t^{-k} \int_{B_t \cap \Sigma} f \, , \tag{1.57}$$

then Proposition 1.11 implies that

$$g'(t) \geq -\frac{\lambda}{2} \, s^{-2} \, t^{1-k} \int_{B_t \cap \Sigma} f = -\frac{\lambda}{2} \, s^{-2} \, t \, g(t) \, . \tag{1.58}$$

We can rewrite (1.58) as

$$\frac{g'(t)}{g(t)} \geq -\frac{\lambda}{2}\, s^{-2}\, t \geq -\frac{\lambda}{2s}\,. \tag{1.59}$$

From (1.59), it is obvious that $e^{\lambda\, t/(2s)}\, g(t)$ is monotone nondecreasing and (1.56) follows immediately. ∎

We get immediately the following mean value inequality for the special case of nonnegative subharmonic functions:

Corollary 1.13 *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, $x_0 \in \mathbb{R}^n$, and f is a nonnegative subharmonic function on Σ; then*

$$s^{-k} \int_{B_s \cap \Sigma} f \tag{1.60}$$

is a nondecreasing function of s. In particular, if $x_0 \in \Sigma$, then for all $s > 0$

$$f(x_0) \leq \frac{\int_{B_s \cap \Sigma} f}{\text{Vol}(B_1 \subset \mathbb{R}^k)\, s^k}\,. \tag{1.61}$$

1.4 The Gauss Map

Let $\Sigma^2 \subset \mathbb{R}^3$ be a surface. The *Gauss map* is a continuous choice of a unit normal $N : \Sigma \to \mathbf{S}^2 \subset \mathbb{R}^3$. Observe that there are two choices of such a map N and $-N$ corresponding to a choice of orientation of Σ. If $\Sigma \subset \mathbb{R}^3$ is the graph of a function $u = u(x, y)$, then, as we have already seen, we can take

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}}\,. \tag{1.62}$$

Using (x, y) as coordinates on the graph, we may express the induced metric g as

$$g_{xx} = (1 + u_x^2)\,, \quad g_{xy} = g_{yx} = u_x u_y\,, \quad g_{yy} = (1 + u_y^2)\,. \tag{1.63}$$

By direct calculation, the eigenvalues of the matrix g are 1 and $(1 + |\nabla u|^2)$. This can also easily be seen geometrically.

By the Gauss equation, the Gauss curvature of the graph of u is given by

$$\begin{aligned} K =& \kappa_1\, \kappa_2 = \frac{\langle N_x, (1,0,u_x)\rangle\, \langle N_y, (0,1,u_y)\rangle - \langle N_x, (0,1,u_y)\rangle^2}{|(1,0,u_x) \times (0,1,u_y)|^2} \\ =& \frac{\langle(-u_{xx}, -u_{yx}, 0), (1,0,u_x)\rangle\, \langle(-u_{xy}, -u_{yy}, 0), (0,1,u_y)\rangle}{(1 + |\nabla u|^2)^2} \\ &- \frac{\langle(-u_{xx}, -u_{xy}, 0), (0,1,u_y)\rangle^2}{(1 + |\nabla u|^2)^2} = \frac{u_{xx}\, u_{yy} - u_{xy}^2}{(1 + |\nabla u|^2)^2}\,. \end{aligned} \tag{1.64}$$

Therefore

$$K\,d\,\mathrm{Area} = \frac{u_{xx}\,u_{yy} - u_{xy}^2}{(1+|\nabla u|^2)^{\frac{3}{2}}}\;dx\wedge dy\,. \tag{1.65}$$

Similarly, we may express the second fundamental form A in the coordinates (x,y) as

(1.66)

$$A_{xx} = \frac{u_{xx}}{(1+|\nabla u|^2)^{\frac{1}{2}}}\,,\quad A_{xy} = A_{yx} = \frac{u_{xy}}{(1+|\nabla u|^2)^{\frac{1}{2}}}\,,\quad A_{yy} = \frac{u_{yy}}{(1+|\nabla u|^2)^{\frac{1}{2}}}\,.$$

This expression for the second fundamental form and the bound on the eigenvalues of g (see (1.63)) together imply

$$\frac{|\,\mathrm{Hess}_u\,|^2}{(1+|\nabla u|^2)^3} \leq |A|^2 \leq 2\frac{|\,\mathrm{Hess}_u\,|^2}{1+|\nabla u|^2}\,. \tag{1.67}$$

Alternatively, we could have computed the Gaussian curvature using moving frames (see, for instance, [Sp]). Namely, if $\Theta = (\Theta_i)_i$ is an orthonormal coframing (that is, $\Theta_i = g(\cdot, E_i)$ where E_i is an orthonormal frame in a neighborhood of x), then the Cartan equations assert that

$$d\Theta = -\omega\wedge\Theta \tag{1.68}$$

and

$$d\omega = -\omega\wedge\omega + \Omega\,. \tag{1.69}$$

Here $\omega = (\omega_{i,j})$ is the skew-symmetric matrix of connection one-forms and $\Omega = (\Omega_{i,j})$ is the matrix of curvature two-forms. In dimension two, the second equation reduces to

$$d\omega = \Omega\,. \tag{1.70}$$

For the graph of u, we let the frame be

$$e_1 = |\nabla u|^{-1}(u_y d/dx - u_x d/dy)\,, \tag{1.71}$$

$$e_2 = |\nabla u|^{-1}(1+|\nabla u|^2)^{-1/2}(u_x d/dx + u_y d/dy)\,, \tag{1.72}$$

and the dual coframe be given by

$$\Theta_1 = |\nabla u|^{-1}(u_y dx - u_x dy)\,, \tag{1.73}$$

$$\Theta_2 = |\nabla u|^{-1}(1+|\nabla u|^2)^{1/2}(u_x dx + u_y dy)\,. \tag{1.74}$$

Hence

(1.75)

$$\omega_{1,2} = -|\nabla u|^{-2}(1+|\nabla u|^2)^{-1/2}((u_x u_{xy} - u_y u_{xx})dx + (u_x u_{yy} - u_y u_{xy})dy)\,.$$

A straightforward calculation shows that for some constant C

$$|\omega_{1,2}| \leq C\,|dN| = C\,|A|\,. \tag{1.76}$$

If Σ is minimal, then the Gauss map is an (anti) conformal map since the eigenvalues of the Weingarten map are κ_1 and $\kappa_2 = -\kappa_1$. Moreover, for a minimal surface

$$|A|^2 = \kappa_1^2 + \kappa_2^2 = -2\,\kappa_1\,\kappa_2 = -2\,K\,, \tag{1.77}$$

and the area of the Gauss map is a multiple of the total curvature. This conformality of the Gauss map for a minimal surface in $\mathbb{R}^3$, namely (1.77), can be used to prove the classical Bernstein theorem described in the next section.

1.5 The Theorem of Bernstein

Before we prove the famous theorem of Bernstein, we will give a bound for the total curvature of a minimal graph. We will later see that, with some more work, this bound can be used to give curvature estimates for minimal graphs. Such curvature estimates were proven originally by Heinz [He] using complex analysis and provided a generalization of the theorem of Bernstein.

Lemma 1.14 *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, then for all nonnegative Lipschitz functions η with support contained in Ω*

$$\int_{\mathrm{Graph}_u} |A|^2\,\eta^2 \leq 16\,C^2 \int_{\mathrm{Graph}_u} |\nabla_{\mathrm{Graph}_u}\eta|^2\,. \tag{1.78}$$

PROOF: Set $\Sigma = \mathrm{Graph}_u$. By minimality, Stokes' theorem, the Cauchy-Schwarz inequality, and (1.76),

$$\begin{aligned}\int_\Sigma \eta^2\,|A|^2 &= -2\int_\Sigma \eta^2\,K = -2\int_\Sigma \eta^2\,\Omega_{1,2}\\ &= -2\int_\Sigma \eta^2\,d\omega_{1,2} = 4\int_\Sigma \eta\,d\eta\wedge\omega_{1,2}\\ &\leq 4\,C\int_\Sigma \eta\,|d\eta|\,|A| \leq 4\,C\left(\int_\Sigma \eta^2\,|A|^2\right)^{\frac{1}{2}}\left(\int_\Sigma |d\eta|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{1.79}$$

Therefore

$$\int_\Sigma \eta^2\,|A|^2 \leq 16\,C^2\int_\Sigma |d\eta|^2\,. \tag{1.80}$$

This proves the lemma. ∎

Corollary 1.15 *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, $\kappa > 1$, and Ω contains a ball of radius $\kappa\,R$ centered at the origin, then*

$$\int_{B_{\sqrt{\kappa}\,R}\cap\mathrm{Graph}_u} |A|^2 \leq \frac{16\,C^2\ \mathrm{e}^2\ \mathrm{Area}(\mathbf{S}^2)}{\log\kappa}\,. \tag{1.81}$$

PROOF: Set $\Sigma = \text{Graph}_u$. Define the cutoff function η on all of $\mathbb{R}^3$ and then restrict it to the graph of u as follows: Let r denote the distance to the origin in $\mathbb{R}^3$. Set $\eta = 1$ for $r^2 \leq \kappa\, R^2$, $\eta = 2 - 2\, \log(r\, R^{-1}) / \log \kappa$ for $\kappa\, R^2 < r^2 \leq R^2$, and $\eta = 0$ for $r^2 > \kappa\, R^2$.

Applying Lemma 1.14 with the cutoff function η and the area bound (1.18), we get

$$\begin{aligned}
\int_{B_{\sqrt{\kappa}\, R} \cap \Sigma} |A|^2 &\leq \int_{\Sigma} \eta^2\, |A|^2 \leq 16\, C^2 \int_{\Sigma} |d\eta|^2 \leq \frac{64\, C^2}{(\log \kappa)^2} \int_{B_{\kappa\, R} \cap \Sigma} r^{-2} \\
&\leq \frac{64\, C^2}{(\log \kappa)^2} \sum_{\ell = (\log \kappa)/2}^{\log \kappa} \int_{(B_{e^{\ell}\, R} \setminus B_{e^{\ell - 1}\, R}) \cap \Sigma} r^{-2} \qquad (1.82) \\
&\leq \frac{64\, C^2}{(\log \kappa)^2} \sum_{\ell = (\log \kappa)/2}^{\log \kappa} e^2 \, \frac{\text{Area}(\mathbf{S}^2)}{2} \leq \frac{16\, C^2\; e^2\; \text{Area}(\mathbf{S}^2)}{\log \kappa}\, .
\end{aligned}$$

∎

The above argument, i.e., integration by parts with this particular choice of η, is often refered to as “a logarithmic cutoff argument.” It is quite useful when the surface has at most quadratic area growth (as above).

As a consequence of this corollary, we get the following theorem of S. Bernstein [Be] from 1916:

Theorem 1.16 (S. Bernstein [Be]) *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation then $u(x, y) = ax + by + c$ for some constants $a,\, b,\, c \in \mathbb{R}$.*

PROOF: (The proof we give here is due to L. Simon [Si5]). By the previous corollary, we have for all $R > 1$

$$\int_{B_{\sqrt{R}} \cap \text{Graph}_u} |A|^2 \leq \frac{16\; e^2\; \text{Area}(\mathbf{S}^2)}{\log R}\, . \qquad (1.83)$$

Letting $R \to \infty$, we conclude that $|A|^2 \equiv 0$; hence $0 = u_{xx} = u_{xy} = u_{yy}$ and therefore $u = ax + by + c$ for some constants a, b, $c \in \mathbb{R}$. ∎

The previous proof of the theorem of Bernstein relied on minimality for two facts: the area bound for minimal graphs, (1.18), and the conformality of the Gauss map, (1.77). This proof can actually be applied to a wider class of differential equations where the conformality is replaced by quasi-conformality. We will briefly return to this later (in (5.7), where we also define quasi-conformality), but we will not discuss estimates for quasi-conformal maps in these notes. A detailed discussion may be found in chapter 16 of [GiTr].

1.6 The Strong Maximum Principle

The following lemma is the local version of the strong maximum principle for minimal hypersurfaces:

Lemma 1.17 *Let $\Omega \subset \mathbb{R}^{n-1}$ be an open connected neighborhood of the origin. If $u_1, u_2 : \Omega \to \mathbb{R}$ are solutions of the minimal surface equation with $u_1 \leq u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$.*

PROOF: Since

$$\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}} - \frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}} \tag{1.84}$$

$$= \frac{\sqrt{1+|\nabla u_2|^2}\,(\nabla u_1 - \nabla u_2) + (\sqrt{1+|\nabla u_2|^2} - \sqrt{1+|\nabla u_1|^2})\,\nabla u_2}{\sqrt{1+|\nabla u_1|^2}\,\sqrt{1+|\nabla u_2|^2}}$$

$$= \frac{\nabla u_1 - \nabla u_2}{\sqrt{1+|\nabla u_1|^2}} + \frac{(|\nabla u_2|^2 - |\nabla u_1|^2)\,\nabla u_2}{(\sqrt{1+|\nabla u_1|^2} + \sqrt{1+|\nabla u_2|^2})\,\sqrt{1+|\nabla u_1|^2}\,\sqrt{1+|\nabla u_2|^2}}$$

and both u_1 and u_2 satisfy the minimal surface equation, we get

(1.85)

$$0 = \operatorname{div}\left(\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}} - \frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}}\right)$$

$$= \operatorname{div}\left(\frac{\nabla (u_1 - u_2)}{\sqrt{1+|\nabla u_1|^2}}\right)$$

$$- \operatorname{div}\left(\frac{\langle \nabla (u_1 - u_2), \nabla (u_1 + u_2)\rangle}{(\sqrt{1+|\nabla u_1|^2} + \sqrt{1+|\nabla u_2|^2})\sqrt{1+|\nabla u_1|^2}\sqrt{1+|\nabla u_2|^2}}\nabla u_2\right).$$

From this, we conclude that $v = u_1 - u_2$ satisfies an equation of the form

$$0 = \operatorname{div}(a_{i,j}\,\nabla v) + b_i\,\nabla v\,. \tag{1.86}$$

Moreover, if $|\nabla u_1|$, $|\nabla u_2|$ are sufficiently small, then $\lambda\,|x|^2 \leq a_{i,j}\,x_i\,x_j$ for some $\lambda > 0$. Therefore the usual strong maximum principle, see, for instance, [HaLi] or theorem 3.5 of [GiTr], gives the claim. ∎

By writing a hypersurface locally as the graph of a function, we see that Lemma 1.17 has the following immediate consequence:

Corollary 1.18 (The Strong Maximum Principle) *If $\Sigma_1, \Sigma_2 \subset \mathbb{R}^n$ are complete connected minimal hypersurfaces (without boundaries), $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, and Σ_2 lies on one side of Σ_1, then $\Sigma_1 = \Sigma_2$.*

1.7 Second Variation Formula, Morse Index, and Stability

Suppose now that $\Sigma^k \subset M^n$ is a minimal submanifold; we want to compute the second derivative of the area functional for a variation of Σ. Therefore, let again F be a variation of Σ with compact support. In fact, we will assume that F is a normal variation, that is, $F_t^T \equiv 0$ on Σ. Let x_i be local coordinates on Σ, set $g_{i,j}(t) = g(F_{x_i}, F_{x_j})$, and set $\nu_t = \sqrt{\det(g_{i,j}(t))}\,\sqrt{\det(g^{i,j}(0))}$. Then

$$\frac{d^2}{dt^2_{t=0}}\operatorname{Vol}(F(\Sigma,t)) = \int \frac{d^2}{dt^2}_{t=0}\nu_t\,\sqrt{\det(g_{ij}(0))}\,. \tag{1.87}$$

Since

$$2\,\frac{d}{dt}\nu_t = \operatorname{Tr}(g'_{ij}(t)g^{\ell m}(t))\,\nu_t\,, \tag{1.88}$$

then

$$2\,\frac{d^2}{dt^2_{t=0}}\nu_t = \operatorname{Tr}(g''_{ij}(0)) - \operatorname{Tr}(g'_{ij}(0)\,g'_{\ell m}(0)) + \frac{1}{2}\,[\operatorname{Tr}(g'_{ij}(0))]^2\,. \tag{1.89}$$

Therefore, since Σ is minimal,

$$2\,\frac{d^2}{dt^2_{t=0}}\nu_t = \operatorname{Tr}(g''_{ij}(0)) - \operatorname{Tr}(g'_{ij}(0)\,g'_{\ell m}(0))\,. \tag{1.90}$$

To evaluate $d^2/dt^2_{t=0}\nu_t$ at some point $x \in \Sigma$, we may choose the coordinate system x_i to be orthonormal at x. An easy calculation gives

$$g'_{ij}(0) = g(F_{x_i t}, F_{x_j}) + g(F_{x_i}, F_{x_j t}) = -2\,g(A(F_{x_i}, F_{x_j}), F_t)\,, \tag{1.91}$$

and at x

(1.92)

$$\begin{aligned}
\sum_{i=1}^k g''_{ii}(0) &= 2\sum_{i=1}^k g(F_{x_i tt}, F_{x_i}) + 2\sum_{i=1}^k g(F_{x_i t}, F_{x_i t}) \\
&= 2\sum_{i=1}^k g(F_{x_i t}, F_{x_i t}) + 2\sum_{i=1}^k g(\mathrm{R}_M(F_t, F_{x_i})F_t, F_{x_i}) + \operatorname{div}_\Sigma(F_{tt}) \\
&= 2\sum_{i=1}^k g(F^T_{x_i t}, F^T_{x_i t}) + 2\sum_{i=1}^k g(F^N_{x_i t}, F^N_{x_i t}) \\
&\quad + 2\sum_{i=1}^k g(\mathrm{R}_M(F_t, F_{x_i})F_t, F_{x_i}) + \operatorname{div}_\Sigma(F_{tt}) \\
&= 2\sum_{i,j=1}^k |g(A(E_i, E_j), F_t)|^2 + 2|\nabla^N_\Sigma F_t|^2 \\
&\quad + 2\sum_{i=1}^k g(\mathrm{R}_M(F_t, E_i)F_t, E_i) + \operatorname{div}_\Sigma(F_{tt})\,.
\end{aligned}$$

Here R_M is the Riemann curvature tensor of M. To get the second equality, we used that at x

$$
\begin{aligned}
\sum_{i=1}^k g(\nabla_{F_t}\nabla_{F_t}F_{x_i}, F_{x_i}) &= \sum_{i=1}^k g(\nabla_{F_t}\nabla_{F_{x_i}}F_t, F_{x_i}) \\
&= \sum_{i=1}^k g(\mathrm{R}_M(F_t, F_{x_i})F_t, F_{x_i}) + \sum_{i=1}^k g(\nabla_{F_{x_i}}\nabla_{F_t}F_t, F_{x_i}) \\
&= \sum_{i=1}^k g(\mathrm{R}_M(F_t, F_{x_i})F_t, F_{x_i}) + \mathrm{div}_\Sigma(F_{tt})\,.
\end{aligned} \tag{1.93}
$$

Therefore, we get at x

$$
\begin{aligned}
\frac{d^2}{dt^2}_{t=0}\nu_t = &- \sum_{i,j=1}^k |g(F_t, A(E_i,E_j))|^2 \\
&+ |\nabla_\Sigma^N F_t|^2 - \sum_{i=1}^k g(\mathrm{R}_M(F_t,E_i)E_i, F_t) + \mathrm{div}_\Sigma(F_{tt})\,.
\end{aligned} \tag{1.94}
$$

Inserting (1.94) into (1.87), integrating and using the minimality of Σ and Stokes' theorem, we get

$$
\begin{aligned}
\frac{d^2}{dt^2}_{t=0}\mathrm{Vol}(F(\Sigma,t)) &= -\sum_{i,j=1}^k \int_\Sigma |g(F_t, A(E_i,E_j))|^2 \\
&\quad + \int_\Sigma |\nabla_\Sigma^N F_t|^2 - \sum_{i=1}^k \int_\Sigma g(\mathrm{R}_M(F_t,E_i)E_i, F_t) \\
&= -\int_\Sigma g(F_t, L\,F_t)\,.
\end{aligned} \tag{1.95}
$$

The self-adjoint operator L is the so-called *stability operator* (or *Jacobi operator*) defined on a normal vector field X to Σ by

$$
L\,X = \Delta_\Sigma^N\,X + \sum_{i=1}^k \mathrm{R}_M(X,E_i)\,E_i + \tilde{A}(X)\,, \tag{1.96}
$$

where $\tilde{A}$ is *Simons' operator* defined by

$$
\tilde{A}(X) = \sum_{i,j=1}^k g(A(E_i,E_j),X)\,A(E_i,E_j) \tag{1.97}
$$

and Δ_Σ^N is the *Laplacian on the normal bundle*, that is,

$$
\Delta_\Sigma^N\,X = \sum_{i=1}^k (\nabla_{E_i}\nabla_{E_i}X)^N - \sum_{i=1}^k (\nabla_{(\nabla_{E_i}E_i)^T}X)^N\,. \tag{1.98}
$$

A normal vector field X with $L\,X = 0$ is said to be a *Jacobi field*.

For a hypersurfaces with a trivial normal bundle, the stability operator simplifies significantly since, in this case, it becomes an operator on functions. Namely, if we identify a normal vector field $X = \eta\,N$ with η, then

$$L\,\eta = \Delta_\Sigma\,\eta + |A|^2\,\eta + \mathrm{Ric}_M(N,N)\,\eta\,. \tag{1.99}$$

We will adopt the convention that λ is a (Dirichlet) eigenvalue of L on $\Omega \subset \Sigma$ if there exists a nontrivial normal vector field X which vanishes on $\partial\Omega$ so that

$$L\,X + \lambda X = 0\,. \tag{1.100}$$

Definition 1.19 The *Morse index* of a compact minimal surface Σ is the number of negative eigenvalues of the stability operator L (counted with multiplicity) acting on the space of smooth sections of the normal bundle which vanish on the boundary.

The second variation formula shows that if $\Sigma^k \subset M^n$ is a minimal submanifold, then the Hessian of the area functional at Σ is given by

$$-\int_\Sigma g(\cdot, L\,\cdot)\,. \tag{1.101}$$

It follows that we could have equivalently defined the Morse index of Σ to be the index of Σ as a critical point for the area functional.

We say that a minimal submanifold $\Sigma^k \subset M^n$ is *stable* if for all variations F with boundary fixed

$$\frac{d^2}{dt^2_{t=0}}\ \mathrm{Vol}(F(\Sigma,t)) = -\int_\Sigma g(F_t, L\,F_t) \geq 0\,. \tag{1.102}$$

Observe that stability is the same as requiring the stability operator to be negative semidefinite (i.e., Morse index zero). Note also that if $\Sigma^{n-1} \subset \mathbb{R}^n$ is the graph of a function satisfying the minimal surface equation, then Σ is stable since Σ is, in fact, area-minimizing. A complete (possibly noncompact) minimal submanifold without boundary is said to be *stable* if all compact subdomains are stable.

For stable minimal hypersurfaces, we have the following useful inequality:

Lemma 1.20 (The Stability Inequality) *Suppose that $\Sigma^{n-1} \subset M^n$ is a stable minimal hypersurface with trivial normal bundle, then for all Lipschitz functions η with compact support*

$$\int_\Sigma (\inf_M\ \mathrm{Ric}_M + |A|^2)\,\eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2\,. \tag{1.103}$$

PROOF: Since Σ is stable,

$$0 \leq -\int_\Sigma \eta\,L\eta = -\int_\Sigma (\eta\,\Delta_\Sigma\,\eta + |A|^2\,\eta^2 + \mathrm{Ric}_M(N,N)\,\eta^2)\,. \tag{1.104}$$

Integrating by parts gives

$$\int_\Sigma (\mathrm{Ric}_M(N,N) + |A|^2)\,\eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2 \,. \tag{1.105}$$

This proves the lemma. ∎

By using Lemma 1.20 and a logarithmic cutoff argument (as in the proof of Theorem 1.16), it is easy to give a second proof of Theorem 1.16. We will return to this point of view in the next chapter.

We will close this section with some useful characterizations of stability for minimal hypersurfaces with trivial normal bundle and we will derive some consequences. This will require more background in PDE than in the rest of these notes; when this occurs, precise references will be given.

For minimal hypersurfaces with trivial normal bundle, we saw that stability was equivalent to $\lambda_1(\Omega, L) \geq 0$ for every $\Omega \subset \Sigma$ where

$$\lambda_1(\Omega, L) = \inf \left\{ -\int \eta\, L\eta \mid \eta \in C_0^\infty(\Omega) \text{ and } \int_\Omega \eta^2 = 1 \right\} . \tag{1.106}$$

For smooth functions u, we define the H^1-norm by $|u|^2_{H^1} = \int u^2 + \int |\nabla u|^2$. The Sobolev space $H^1_0(\Omega)$ is the closure of the compactly supported smooth functions on Ω with respect to the H^1-norm. Similarly, $H^1(\Omega)$ is the closure of the space of smooth functions on Ω with respect to the H^1-norm. By standard elliptic theory, see, for instance, [HaLi] or [GiTr], we get the following:

Lemma 1.21 *Let L and $\Omega \subset \Sigma$ be as above. If $u \in H^1_0(\Omega)$ satisfies $\int_\Omega u^2 = 1$ and $-\int_\Omega u\, Lu = \lambda_1 = \lambda_1(\Omega, L)$, then $Lu = -\lambda_1 u$.*

PROOF: If ψ is a smooth function with compact support in Ω and

$$\int \psi u = 0\,, \tag{1.107}$$

then obviously

$$\frac{d}{dt}_{t=0} \int (u + t\psi)^2 = 0\,. \tag{1.108}$$

By the definition of λ_1, (1.108) implies that

$$0 = \frac{d}{dt}_{t=0} \int (u + t\psi)\, L(u + t\psi) = 2 \int \psi\, Lu\,, \tag{1.109}$$

where the second equality follows from Stokes' theorem. By approximation, equation (1.109) holds for any $\psi \in H^1_0(\Omega)$ satisfying (1.107). In particular, given any $\phi \in H^1_0(\Omega)$, then (1.109) holds for $\psi = \phi - u \int \phi u$ and thus

$$\int \phi\, Lu = \int \phi u \int u\, Lu = -\lambda_1 \int \phi u\,. \tag{1.110}$$

Since (1.110) holds for all $\phi \in H^1_0(\Omega)$, u is a weak solution to $Lu = -\lambda_1\, u$. The lemma now follows by elliptic regularity (theorem 8.8 of [GiTr]). ∎

We will next recall the Harnack inequality for nonnegative solutions of uniformly elliptic equations. The version that we will use is contained in theorem 8.20 of [GiTr] and applies to a very general class of operators. For the next lemma, let $\mathcal{L}$ be a second-order linear differential operator on $\mathbb{R}^n$ given by

$$\mathcal{L}\, u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a^{ij}(x) \frac{\partial u}{\partial x_j} + b^i(x)\, u \right) + \sum_{i=1}^{n} c^i(x)\, \frac{\partial u}{\partial x_i} + d(x)\, u \,, \tag{1.111}$$

where the coefficients a^{ij}, b^i, c^i, d are measurable functions.

Lemma 1.22 *Let $\mathcal{L}$ be a second-order linear differential operator on $\Omega \subset \mathbb{R}^n$ with bounded measurable coefficients a^{ij}, b^i, c^i, d as in* (1.111) *satisfying*

$$\sum_{i,j=1}^{n} a^{ij} x_i x_j \geq \lambda |x|^2 \tag{1.112}$$

for some $\lambda > 0$ and

$$\sum_{i,j=1}^{n} (a^{ij})^2 \leq \Lambda \,, \tag{1.113}$$

$$\lambda^{-2} \sum_{i=1}^{n} (|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1} |d(x)| \leq \nu^2 \,, \tag{1.114}$$

for some $\Lambda, \nu < \infty$. Suppose that $u \in C^0(\Omega) \cap H^1(\Omega)$ satisfies $u \geq 0$ in Ω and $\mathcal{L}u = 0$ weakly in Ω. Then, for any ball $B_{4R}(y) \subset \Omega$, we have

$$\sup_{B_R(y)} u \leq C \inf_{B_R(y)} u \,, \tag{1.115}$$

where $C = C(n, \frac{\Lambda}{\lambda}, \nu R) < \infty$.

By using local coordinates and a covering argument with chains of balls we can extend the Harnack inequality of Lemma 1.22 to elliptic equations on bounded domains in a Riemannian manifold.

Combining Lemma 1.21 and the Harnack inequality, we see in the next lemma that any eigenfunction for the first eigenvalue cannot change sign.

Lemma 1.23 *If u is a smooth function on Ω that vanishes on $\partial\Omega$ and $Lu = -\lambda_1 u$ where $\lambda_1 = \lambda_1(\Omega, L)$, then u cannot change sign in Ω.*

PROOF: We may assume that u is not identically zero. Since u vanishes on $\partial\Omega$, so does $|u|$. In fact, it is easy to see that $|u|$ also achieves the minimum in (1.106) and hence, by Lemma 1.21, we have $L\,|u| = -\lambda_1\,|u|$. Since $|u| \geq 0$ and $|u|$ is not identically zero, the Harnack inequality, Lemma 1.22, implies that $|u| > 0$ in Ω and the lemma follows. ∎

Since the eigenfunctions are all orthogonal to each other, Lemma 1.23 implies that only the lowest eigenfunction does not change sign and, in fact, the first eigenvalue has multiplicity one. As a consequence, we see that if $\Sigma \subset M$ is a stable minimal hypersurface with trivial normal bundle and without boundary, then $\tilde{\Sigma} \subset \tilde{M}$ is also stable where $G : \tilde{M} \to M$ is a covering map, $\tilde{\Sigma} = G^{-1}(\Sigma)$, and $\tilde{M}$ is given the pullback metric. On the other hand, easy examples show that a cover of a stable minimal submanifold is not in general stable (consider, for instance, $\mathbb{RP}^2 \subset \mathbb{RP}^3$).

More generally, we have the following:

Lemma 1.24 *Let Σ be a minimal hypersurface with trivial normal bundle, L its stability operator, and $\Omega \subset \Sigma$ a bounded domain. If there exists a positive function u on Ω with $Lu = 0$, then Ω is stable.*

PROOF: Set $q = |A|^2 + \mathrm{Ric}_M(N, N)$ so that $L = \Delta_\Sigma + q$. Since $u > 0$, $w = \log u$ is well-defined and satisfies

$$\Delta_\Sigma w = -q - |\nabla_\Sigma w|^2 . \tag{1.116}$$

Let f be a compactly supported smooth function on Ω. Multiplying both sides of (1.116) by f^2 and integrating by parts gives

$$\begin{aligned} \int f^2\, q + \int f^2\, |\nabla_\Sigma w|^2 &= -\int f^2\, \Delta_\Sigma w \leq 2 \int |f|\, |\nabla_\Sigma f|\, |\nabla_\Sigma w| \\ &\leq \int f^2\, |\nabla_\Sigma w|^2 + \int |\nabla_\Sigma f|^2 , \end{aligned} \tag{1.117}$$

where the second inequality follows from the Cauchy-Schwarz inequality. Cancelling the $\int f^2\, |\nabla_\Sigma w|^2$ term from both sides of (1.117), we see that $-\int f\, Lf \geq 0$. Since this is true for any such f, the lemma follows. ∎

Note that if Σ is closed or, more generally, if u vanishes on $\partial\Sigma$, then Lemma 1.24 follows immediately from Lemma 1.23.

The variation $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ given by $F(\cdot, t) : (x_1, x_2, x_3) \to (x_1, x_2, x_3 + t)$ is a one-parameter group of isometries of $\mathbb{R}^3$, and hence for any surface $\Sigma \subset \mathbb{R}^3$ we have that $\mathrm{Area}\ F(\Sigma, t)$ is constant. The variation vector field is $F_t = (0, 0, 1)$. If Σ is minimal, then the second variation formula implies that $\langle N, (0, 0, 1)\rangle$ (the normal component of the variation vector field) is a Jacobi field. Furthermore, when $\Sigma = \mathrm{Graph}_u$

$$\langle N, (0, 0, 1)\rangle = \frac{1}{\sqrt{1 + |\nabla_{\mathbb{R}^2} u|^2}} , \tag{1.118}$$

is therefore a positive Jacobi field. Consequently, Lemma 1.24 gives another way to see that minimal graphs are stable.

A manifold Σ is said to be *parabolic* if any positive superharmonic function u (i.e., $\Delta_\Sigma u \leq 0$) is constant. The next proposition shows that any Σ with quadratic volume growth is parabolic. In this proposition, we will let $B_s^\Sigma = B_s^\Sigma(p)$ denote an intrinsic (geodesic) ball in Σ.

Proposition 1.25 *Any complete surface* Σ *with* $\text{Area}\,(B_s^{\Sigma}) \leq C\, s^2$ *for all* $s > 0$ *is parabolic.*

PROOF: Suppose that $u > 0$ and $\Delta_{\Sigma} u \leq 0$ and define $w = \log u$ so that $|\nabla_{\Sigma} w|^2 \leq -\Delta_{\Sigma} w$. Let r denote the distance to p and define the cutoff function η by $\eta = 1$ for $r^2 \leq R$, $\eta = 2 - 2\,\frac{\log r}{\log R}$ for $R < r^2 \leq R^2$, and set $\eta = 0$ for $r^2 > R^2$. By Stokes' theorem and the absorbing inequality ($2ab \leq \frac{1}{2} a^2 + 2b^2$),

$$\begin{aligned} \int \eta^2\, |\nabla_{\Sigma} w|^2 &\leq - \int \eta^2\, \Delta_{\Sigma} w \leq 2 \int \eta\, |\nabla_{\Sigma} \eta|\, |\nabla_{\Sigma} w| \\ &\leq \frac{1}{2} \int \eta^2\, |\nabla_{\Sigma} w|^2 + 2 \int |\nabla_{\Sigma} \eta|^2 \,. \end{aligned} \tag{1.119}$$

Substituting the definition of η and the area bound gives

$$\begin{aligned} \int_{B^{\Sigma}_{\sqrt{R}}} |\nabla_{\Sigma} w|^2 &\leq \int \eta^2\, |\nabla_{\Sigma} w|^2 \leq 4 \int |\nabla_{\Sigma} \eta|^2 \\ &\leq \frac{16}{(\log R)^2} \sum_{\ell = \frac{1}{2}\log R}^{\log R} \int_{(B^{\Sigma}_{e^{\ell}} \setminus B^{\Sigma}_{e^{\ell - 1}})} r^{-2} \\ &\leq \frac{16}{(\log R)^2} \sum_{\ell = \frac{1}{2}\log R}^{\log R} C\, e^2 \leq \frac{8\, C\, e^2}{\log R} \,. \end{aligned} \tag{1.120}$$

Letting $R \to \infty$, we get that w is constant. ∎

Applying Proposition 1.25, we see that an entire minimal graph Σ^2 is parabolic. This follows since the intrinsic distance is bounded from below by the Euclidean distance and therefore the area bound (1.18) implies that minimal graphs have quadratic area growth. Setting $u = \langle N, (0,0,1) \rangle$ as in (1.118) gives a positive Jacobi field. In particular,

$$\Delta_{\Sigma} u = -(|A|^2 + \text{Ric}_{\mathbb{R}^3}(N,N))\, u = -\,|A|^2\, u \leq 0\,, \tag{1.121}$$

so that u is a positive superharmonic function. By Proposition 1.25, u must be constant so that $\Delta_{\Sigma} u = 0$ and hence $|A|^2 = 0$. In other words, any complete minimal graph defined on $\mathbb{R}^2$ must be flat. This yields another proof of the Bernstein theorem, Theorem 1.16.

We will next give a characterization of stability for complete noncompact minimal hypersurfaces with trivial normal bundle. We will assume that the boundary is smooth if it is nonempty.

Proposition 1.26 (Fischer-Colbrie and Schoen [FiSc]) *If* Σ *is a complete noncompact minimal hypersurface with trivial normal bundle, then the following are equivalent:*

(1.122) $\lambda_1(\Omega, L) \geq 0$ *for every bounded domain* $\Omega \subset \Sigma$.

(1.123) $\lambda_1(\Omega, L) > 0$ *for every bounded domain* $\Omega \subset \Sigma$.

(1.124) *There exists a positive function* u *with* $L\, u = 0$.

PROOF: By Lemma 1.24, (1.124) implies (1.122).

Clearly (1.123) implies (1.122). To see the equivalence of (1.122) and (1.123), given any bounded domain $\Omega_0 \subset \Sigma$ choose a strictly larger bounded domain Ω_1. The variational characterization of eigenvalues, (1.106), implies that

$$\lambda_1(\Omega_0, L) \geq \lambda_1(\Omega_1, L) \geq 0\,, \tag{1.125}$$

where the second inequality follows from (1.122). Let u_0 denote the first eigenfunction for L on Ω_0, and define u_1 on Ω_1 by

$$u(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega_0\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.126}$$

If we had equality in (1.125), then, by Lemma 1.21, $L\,u_1 = -\lambda_1 u_1$ on Ω_1 and, by Lemma 1.23, $u_1 > 0$ on Ω_1. This is not possible since u_1 vanishes on $\Omega_1 \setminus \Omega_0$, and thus the equivalence of (1.122) and (1.123) follows.

It remains to show that (1.123) implies (1.124). To do this, fix $p \in \Sigma$ and for each $r > 0$ let

$$B_r^\Sigma = B_r^\Sigma(p) = \{q \in \Sigma \mid \text{dist}_\Sigma(p,q) < r\}\,. \tag{1.127}$$

Since $\lambda_1(B_r^\Sigma, L) > 0$, by the Fredholm alternative (see theorem 6.15 of [GiTr]), there exists a unique function v_r with

$$L\,v_r = -|A|^2 - \text{Ric}_M(N,N) \text{ on } B_r^\Sigma \quad \text{and} \quad v_r = 0 \text{ on } \partial B_r^\Sigma\,. \tag{1.128}$$

Setting $u_r = v_r + 1$, (1.128) gives

$$L\,u_r = 0 \text{ on } B_r^\Sigma \quad \text{and} \quad u_r = 1 \text{ on } \partial B_r^\Sigma\,. \tag{1.129}$$

We claim that $u_r > 0$ in B_r^Σ. By the Harnack inequality, Lemma 1.22, it suffices to show that $u_r \geq 0$ in B_r^Σ. If this fails, then we can choose a nonempty connected component Ω of the open set $\{x \in B_r^\Sigma \mid u_r(x) < 0\}$. By construction, we have $u_r < 0$ in Ω and $u_r = 0$ on $\partial\Omega$, and Lemma 1.23 implies that $\lambda_1(\Omega, L) = 0$. This gives a contradiction and hence $u_r > 0$ in B_r^Σ. Define $w_r > 0$ by

$$w_r = (u_r(p))^{-1}\,u_r \tag{1.130}$$

and observe that $Lw_r = 0$ and $w_r(p) = 1$.

Now, let K be any compact set with $K \subset B_{R_0}^\Sigma$. Applying the Harnack inequality (see theorem 8.27 of [GiTr] for the estimates up to $\partial\Sigma$), we get for any $r \geq 2R_0$ that

$$\sup_K w_r \leq C_K\,. \tag{1.131}$$

The interior and boundary Schauder estimates (theorems 6.2 and 6.6 of [GiTr]) imply that

$$|w_r|_{C_K^{2,\alpha}} \leq C'_K\,. \tag{1.132}$$

In other words, if $K \subset B_{R_0^\Sigma}$, we have uniform $C^{2,\alpha}_K$ estimates for every w_r for $r \geq 2R_0$. By the Arzela-Ascoli theorem, we can choose a subsequence of the w_r that converges uniformly in $C^{2,\frac{\alpha}{2}}$ on compact sets to a function w. This convergence guarantees that w satisfies $Lw = 0$. Since each w_r was positive and $w_r(p) = 1$, w is nonnegative and has $w(p) = 1$. Finally, the Harnack inequality implies that w is also positive, which completes the proof. ∎

We can use Proposition 1.26 to give a slight generalization of the Bernstein theorem.

Corollary 1.27 *If $\Sigma \subset \mathbb{R}^3$ is a complete, connected, stable, parabolic, orientable minimal surface without boundary, then it must be a plane.*

PROOF: Since Σ is orientable and stable, Proposition 1.26 implies that there exists a function $u > 0$ with

$$\Delta_\Sigma u = -(|A|^2 + \mathrm{Ric}_{\mathbb{R}^3}(N,N))\, u = -\,|A|^2\, u \leq 0\,. \tag{1.133}$$

Since Σ is parabolic, u must be constant. Hence (1.133) implies that $|A| \equiv 0$ and the corollary follows. ∎

2

Curvature Estimates and Consequences

In this chapter, we will give various generalizations of the Bernstein theorem discussed in Chapter 1. We begin by deriving Simons' inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbb{R}^n$. In the later sections, we will discuss various applications of such an inequality. Our first application is to a theorem of Choi-Schoen giving curvature estimates for minimal surfaces with small total curvature. Using this estimate, we give a short proof of Heinz's curvature estimate for minimal graphs. Next, we discuss a priori estimates for stable minimal surfaces in three-manifolds, including estimates on area and total curvature of Colding-Minicozzi and the curvature estimate of Schoen. After that, we follow Schoen-Simon-Yau and combine Simons' inequality with the stability inequality to show higher L^p bounds for the norm squared of the second fundamental form for stable minimal hypersurfaces. The higher L^p bounds are then used together with Simons' inequality to show curvature estimates for stable minimal hypersurfaces and to give a generalization due to De Giorgi, Almgren, and Simons of the Bernstein theorem proven in Chapter 1. We close the chapter with a discussion of minimal cones in Euclidean space and the counterexample of Bombieri-De Giorgi-Giusti to the Bernstein theorem in dimension greater than seven.

2.1 Simons' Inequality

In this section, we will derive a very useful differential inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbb{R}^n$. In the later sections of this chapter, we will discuss various applications of such an inequality. This inequality, originally due to J. Simons [Sim], asserts in its most general form that for a minimal hypersurface $\Sigma^{n-1} \subset M^n$

$$\Delta_\Sigma |A|^2 \geq -C\,(1 + |A|^2)^2\,, \tag{2.1}$$

where C depends on the curvature of M and its covariant derivative.

We will now derive Simons' inequality in its original form, see [ScSiYa] for the more general inequality described above.

Lemma 2.1 (Simons' Inequality [Sim]) *Suppose that $\Sigma^{n-1} \subset \mathbb{R}^n$ is a minimal hypersurface; then*

$$\Delta_\Sigma |A|^2 \geq -2\,|A|^4 + 2\,(1 + \frac{2}{n-1})\,|\nabla_\Sigma |A||^2 \,. \tag{2.2}$$

Note that this can equivalently be expressed as

$$|A|\,\Delta_\Sigma\,|A| + \,|A|^4 \geq \frac{2}{n-1}\,|\nabla_\Sigma |A||^2 \,. \tag{2.3}$$

PROOF: Let E_i for $i = 1, \ldots, n$ be a locally defined orthonormal frame in a neighborhood of some $x \in \Sigma$ such that E_n is normal to Σ. Let a be the symmetric two-tensor on Σ given by

$$a(X,Y) = g(A(X,Y), E_n) = g(\nabla_X Y, E_n) = -g(\nabla_X E_n, Y)\,, \tag{2.4}$$

and set $a_{ij} = -g(\nabla_{E_i} E_n, E_j)$. That is,

$$a = \sum_{i,j=1}^{n-1} a_{ij}\,\Theta_i \otimes \Theta_j \,, \tag{2.5}$$

where Θ_i is the dual (to E_i) orthonormal frame. Let $a_{\cdot\cdot,k}$ and $a_{ij,k}$ be defined by

$$a_{\cdot\cdot,k} = \sum_{i,j=1}^{n-1} a_{ij,k}\,\Theta_i \otimes \Theta_j = \nabla_{E_k} a \,. \tag{2.6}$$

Since a is a symmetric two-tensor, it follows that $a_{\cdot\cdot,k}$ is also a symmetric two-tensor. Note that, since the curvature of $\mathbb{R}^n$ vanishes,

$$\begin{aligned} a_{ij,k} &= \nabla^T_{E_k}\, a(E_i, E_j) \\ &= E_k\, a(E_i,E_j) - a(\nabla^T_{E_k} E_i, E_j) - a(E_i, \nabla^T_{E_k} E_j) \\ &= -E_k\, g(\nabla_{E_i} E_n, E_j) + g(\nabla_{\nabla^T_{E_k} E_i} E_n, E_j) + g(\nabla_{E_i} E_n, \nabla_{E_k} E_j) \\ &= -g(\nabla_{E_k} \nabla_{E_i} E_n, E_j) + g(\nabla_{\nabla^T_{E_k} E_i} E_n, E_j) \\ &= -g(\nabla_{E_i} \nabla_{E_k} E_n, E_j) + g(\nabla_{\nabla^T_{E_i} E_k} E_n, E_j) = a_{kj,i} \,. \end{aligned} \tag{2.7}$$

Therefore, let $a_{\cdot\cdot,\cdot}$ be the symmetric three-tensor given by

$$a_{\cdot\cdot,\cdot} = \sum_{i,j,k=1}^{n-1} a_{ij,k}\,\Theta_i \otimes \Theta_j \otimes \Theta_k \,. \tag{2.8}$$

Next, we define a symmetric three-tensor $a_{\cdot\cdot,\cdot\ell}$ by

$$a_{\cdot\cdot,\cdot\ell} = \sum_{i,j,k=1}^{n-1} a_{ij,k\ell}\,\Theta_i \otimes \Theta_j \otimes \Theta_k = \nabla_{E_\ell} a_{\cdot\cdot,\cdot} \,. \tag{2.9}$$

Hence,

$$
\begin{aligned}
a_{ij,k\ell} &= \nabla^T_{E_\ell} a_{\cdot\cdot,\cdot}(E_i, E_j, E_k) \\
&= E_\ell\, a_{\cdot\cdot,\cdot}(E_i, E_j, E_k) - a_{\cdot\cdot,\cdot}(\nabla^T_{E_\ell} E_i, E_j, E_k) \\
&\quad - a_{\cdot\cdot,\cdot}(E_i, \nabla^T_{E_\ell} E_j, E_k) - a_{\cdot\cdot,\cdot}(E_i, E_j, \nabla^T_{E_\ell} E_k) \\
&= E_\ell\, a_{\cdot\cdot,k}(E_i, E_j) - a_{\cdot\cdot,k}(\nabla^T_{E_\ell} E_i, E_j) \\
&\quad - a_{\cdot\cdot,k}(E_i, \nabla^T_{E_\ell} E_j) - \nabla^T_{\nabla^T_{E_\ell} E_k} a(E_i, E_j) \\
&= E_\ell\, E_k\, a(E_i, E_j) - E_\ell\, a(\nabla^T_{E_k} E_i, E_j) - E_\ell\, a(E_i, \nabla^T_{E_k} E_j) \\
&\quad - E_k\, a(\nabla^T_{E_\ell} E_i, E_j) + a(\nabla^T_{E_k} \nabla^T_{E_\ell} E_i, E_j) + a(\nabla^T_{E_\ell} E_i, \nabla^T_{E_k} E_j) \\
&\quad - E_k\, a(E_i, \nabla^T_{E_\ell} E_j) + a(\nabla^T_{E_k} E_i, \nabla^T_{E_\ell} E_j) + a(E_i, \nabla^T_{E_k} \nabla^T_{E_\ell} E_j) \\
&\quad - \nabla^T_{E_\ell} E_k\, a(E_i, E_j) + a(\nabla^T_{\nabla^T_{E_\ell} E_k} E_i, E_j) + a(E_i, \nabla^T_{\nabla^T_{E_\ell} E_k} E_j) \\
&= a_{ij,\ell k} + \sum_{m=1}^{n-1} \mathrm{R}_{k\ell i m}\; a_{mj} + \sum_{m=1}^{n-1} \mathrm{R}_{k\ell j m}\; a_{mi}\,.
\end{aligned} \tag{2.10}
$$

Note that this computation holds for any symmetric two-tensor a. Recall that, by the Gauss equations,

$$
\mathrm{R}_{ijk\ell} = a_{jk}\, a_{i\ell} - a_{ik}\, a_{j\ell}\,, \tag{2.11}
$$

and therefore (2.10) can be written as

(2.12)

$$
\begin{aligned}
a_{ik,jk} &= a_{ik,kj} + \sum_{m=1}^{n-1} \mathrm{R}_{jkim}\; a_{mk} + \sum_{m=1}^{n-1} \mathrm{R}_{jkkm}\; a_{mi} \\
&= a_{ik,kj} + \sum_{m=1}^{n-1} (a_{ki}\, a_{jm} - a_{ji}\, a_{km})\, a_{mk} + \sum_{m=1}^{n-1} (a_{kk}\, a_{jm} - a_{jk}\, a_{km})\, a_{mi}\,.
\end{aligned}
$$

Finally, we are ready to show Simons' equation. Using (2.12), we get that

$$
\begin{aligned}
\Delta_\Sigma |A|^2 &= 2 \sum_{i,j=1}^{n-1} a_{ij}\, \Delta_\Sigma a_{ij} + 2 \sum_{i,j=1}^{n-1} |\nabla_\Sigma a_{ij}|^2 \\
&= 2 \sum_{i,j,k=1}^{n-1} a_{ij}\, a_{ij,kk} + 2 \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \\
&= 2 \sum_{i,j,k=1}^{n-1} a_{ij}\, a_{ik,jk} + 2 \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \\
&= 2 \sum_{i,j,k=1}^{n-1} a_{ij}\, a_{kk,ij} + 2 \sum_{i,j,k,m=1}^{n-1} a_{ij}\,(a_{ki}\, a_{jm} - a_{ji}\, a_{km})\, a_{mk} \\
&\quad + 2 \sum_{i,j,k,m=1}^{n-1} a_{ij}\,(a_{kk}\, a_{jm} - a_{jk}\, a_{km})\, a_{mi} + 2 \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \\
&= -2 \sum_{i,j,k,m=1}^{n-1} a_{ij}^2\, a_{km}^2 + 2 \sum_{i,j,k=1}^{n-1} a_{ij,k}^2\,.
\end{aligned}
\tag{2.13}
$$

Therefore, we get

$$
\Delta_\Sigma |A|^2 = -2\,|A|^4 + 2 \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \tag{2.14}
$$

which is *Simons' equation*.

We will now see how to obtain Simons' inequality from this. Observe first that, since a is symmetric, we may choose E_i, $i = 1, \dots, n-1$, such that at x we have $a_{ij} = \lambda_i\, \delta_{ij}$. Therefore, by the Cauchy-Schwarz inequality

$$
\begin{aligned}
4\,|A|^2\, |\nabla |A||^2 = |\nabla |A|^2|^2 &= \sum_{k=1}^{n-1} [(\sum_{i,j=1}^{n-1} a_{ij}^2)_k]^2 \\
&= 4 \sum_{k=1}^{n-1} (\sum_{i=1}^{n-1} a_{ii,k}\, a_{ii})^2 \leq 4\,|A|^2 \sum_{i,k=1}^{n-1} a_{ii,k}^2\,.
\end{aligned}
\tag{2.15}
$$

Hence (2.15) gives

$$
|\nabla |A||^2 \leq \sum_{i,k=1}^{n-1} a_{ii,k}^2\,. \tag{2.16}
$$

Therefore, by minimality and (2.16)

$$
\begin{aligned}
|\nabla |A||^2 &\leq \sum_{i,k=1}^{n-1} a_{ii,k}^2 = \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^{n-1} a_{ii,i}^2 \\
&= \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^{n-1} \left(\sum_{i\neq j} a_{jj,i} \right)^2 \qquad (2.17) \\
&\leq \sum_{i\neq k} a_{ii,k}^2 + (n-2) \sum_{i=1}^{n-1} \sum_{i\neq j} a_{jj,i}^2 = (n-1) \sum_{i\neq k} a_{ii,k}^2 \\
&= (n-1) \sum_{i\neq k} a_{ik,i}^2 = \frac{n-1}{2} \left(\sum_{i\neq k} a_{ik,i}^2 + \sum_{i\neq k} a_{ki,i}^2 \right) .
\end{aligned}
$$

From (2.16) and (2.17), we get

$$
\begin{aligned}
(1 + \frac{2}{n-1}) \, |\nabla |A||^2 &\leq \sum_{i,k=1}^{n-1} a_{ii,k}^2 + \sum_{i\neq k} a_{ik,i}^2 + \sum_{i\neq k} a_{ki,i}^2 \\
&\leq \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 . \qquad (2.18)
\end{aligned}
$$

Combining (2.18) with Simons' equation (i.e., (2.14)) yields Simons' inequality. ∎

The following is a consequence of this lemma: If $n = 3$ and $\Sigma^2 \subset \mathbb{R}^3$ is a minimal surface, then

$$\Delta \log |A|^2 = \frac{\Delta |A|^2 - 4 \, |\nabla |A||^2}{|A|^2} = -2 \, |A|^2 . \qquad (2.19)$$

This follows since, for a surface, Simons' equation actually implies the equation

$$\Delta |A|^2 = -2 \, |A|^4 + 4 \, |\nabla |A||^2 \qquad (2.20)$$

and not just an inequality. This is easily seen since, in the case of a surface, the Cauchy-Schwarz inequalities applied above are equalities by minimality.

Equation (2.19) has the nice geometric interpretation that

$$|A| \, \langle \cdot , \cdot \rangle \qquad (2.21)$$

is a flat (possibly singular) metric on Σ. Namely, if (Σ, g) is a Riemann surface and f is a positive function on Σ, then the curvature of $(\Sigma, f^2 \, g)$ is given by

$$\mathrm{K}_g = \Delta_g \log f + f^2 \, \mathrm{K}_{f^2 \, g} \, . \qquad (2.22)$$

Taking $f = |A|^{\frac{1}{2}}$ in this formula, where $|A| > 0$, and using Simons' equation in dimension two yields, by minimality,

$$
\begin{aligned}
-|A|^2 = 2\mathrm{K}_\Sigma &= \Delta_\Sigma \log |A| + 2\,|A|\,\mathrm{K}_{|A|\,\Sigma} \\
&= -|A|^2 + 2\,|A|\,\mathrm{K}_{|A|\,\Sigma}\,.
\end{aligned} \tag{2.23}
$$

This clearly implies that $|A|\,\langle\cdot,\cdot\rangle$ is a flat metric on Σ where $|A| > 0$.

The formula for the curvature of the conformally changed metric (i.e., formula (2.22)) can easily be proven using moving frames. Namely, for $x \in \Sigma$, let E_1 and E_2 be an orthonormal frame of (Σ, g) in a neighborhood x and let Θ_i be the dual orthonormal coframe. Then $\bar{\Theta}_i = f\,\Theta_i$ is an orthonormal coframe for $(\Sigma, f^2\,g)$. Further from the Cartan equations we see that

$$
\bar{\omega}_{1,2} = \omega_{1,2} - E_2(\log f)\,\Theta_1 + E_1(\log f)\,\Theta_2 \tag{2.24}
$$

and therefore

$$
\begin{aligned}
\bar{\Omega}_{1,2} = d\bar{\omega}_{1,2} &= d\omega_{1,2} + [E_1(E_1(\log f)) + E_2(E_2(\log f))]\,\Theta_1 \wedge \Theta_2 \\
&\quad - E_2(\log f)\,d\Theta_1 + E_1(\log f)\,d\Theta_2 \\
&= \Omega_{1,2} + \frac{\Delta_g \log f}{f^2}\,\bar{\Theta}_1 \wedge \bar{\Theta}_2\,,
\end{aligned} \tag{2.25}
$$

and the claim follows.

2.2 Curvature Estimates for Minimal Surfaces

In this section, we will give some curvature estimates for minimal surfaces. In particular, we will obtain a priori interior curvature bounds for minimal surfaces which either have small total curvature, are graphical, or have small excess. Before doing so, it will be useful to have the following elementary lemma:

Lemma 2.2 (Small Curvature Implies Graphical) *Suppose that $\Sigma^2 \subset B_{4s} \subset \mathbb{R}^n$ is an immersed surface with $\partial\Sigma \subset \partial B_{4s}$ and*

$$
16\,s^2 \sup_\Sigma |A_\Sigma|^2 \leq 1\,. \tag{2.26}
$$

If $x \in B_{2s} \cap \Sigma$, then the connected component of $B_s(x) \cap \Sigma$ containing x can be written as a graph of a function u over its tangent plane at x with $|\nabla u| \leq 1$ and $\sqrt{2}\,s\,|\,\mathrm{Hess}_u\,| \leq 1$.

PROOF: Within this lemma, we define

$$
d_{x,y} \equiv \mathrm{dist}_{\mathbf{S}^{n-1}}\,(N(x), N(y))\,. \tag{2.27}
$$

In order to show that $x \in B_s \cap \Sigma$ is graphical over its tangent plane at x, it suffices to show that for any $y \in B_s(x) \cap \Sigma$ we have

$$
d_{x,y} < \frac{\pi}{2}\,. \tag{2.28}
$$

If (2.28) holds and we write $B_s(x) \cap \Sigma$ as the graph over $T_x\Sigma$ of a function u then (compare (1.3))

$$1 + |\nabla u|^2 = \langle N(x), N(y) \rangle^{-2} = \cos^{-2} d_{x,y} \, . \tag{2.29}$$

Equation (2.29) implies that so long as

$$d_{x,y} < \frac{\pi}{4} \, ; \tag{2.30}$$

then $|\nabla u| \leq 1$.

Recall that $|\nabla N| \leq |A|$. Therefore, given $y \in \Sigma$, integrating (2.26) along the geodesic joining x and y gives

$$d_{x,y} \leq \frac{1}{4\,s} \; \text{dist}_\Sigma \, (x, y) \, . \tag{2.31}$$

In particular, $\tilde{\Sigma} = \{ y \in \Sigma \mid \text{dist}_\Sigma(x,y) \leq 2\,s \} \subset B_{4s}$ satisfies

$$\sup_{y \in \tilde{\Sigma}} d_{x,y} \leq \frac{1}{2} < \frac{\pi}{4} \, , \tag{2.32}$$

and therefore, since (2.30) holds, $\tilde{\Sigma}$ is a graph over $T_x\Sigma$ with gradient bounded by 1.

We must show that the component of $B_s(x) \cap \Sigma$ containing x is contained in $\tilde{\Sigma}$. Letting $r(y) = |y - x|$, condition (2.30) implies that $|\nabla_\Sigma r| \geq 1/\sqrt{2}$. Consequently, $\tilde{\Sigma}$ must, in fact, contain the component of Σ in an extrinsic ball of radius $\sqrt{2}s$. This completes the proof of the graphical representation and the gradient bound.

Finally, applying the estimate (1.67)

$$|\operatorname{Hess}_u|^2 \leq (1 + |\nabla u|^2)^2 \, |A|^3 \leq 8 \frac{1}{16\, s^2} \, , \tag{2.33}$$

we get the desired bound on the Hessian of u. ∎

Note that, if we replace the bound 1 on the right-hand side of (2.26) by some $\delta < 1$, then instead of (2.32) we get

$$\sup_{y \in \tilde{\Sigma}} d_{x,y} \leq \frac{\delta}{2} \, . \tag{2.34}$$

If $0 < s < \frac{\pi}{4}$, then

$$\frac{\partial}{\partial s} \cos^{-2} s = 2 \, \sin s \, \cos^{-3} s \leq 4 \, , \tag{2.35}$$

so that $\cos^{-2} s \leq 1 + 4\,s$. Therefore, (2.29) and (2.34) imply that

$$|\nabla u|^2 \leq 4\,\delta \, . \tag{2.36}$$

We will next prove the "small total curvature" estimate of Choi and Schoen [CiSc], cf. M. T. Anderson [An] and B. White [Wh1].

Theorem 2.3 (Choi-Schoen [CiSc]) *Let M^n be an n-dimensional Riemannian manifold. There exist, $\epsilon = \epsilon(M) > 0$ and $\rho = \rho(M) > 0$ such that if $r_0 < \rho$, $\Sigma^2 \subset M$ is a compact minimal surface with $\partial\Sigma \subset \partial B_{r_0}(x)$, $0 < \delta \leq 1$, and*

$$\int_{B_{r_0} \cap \Sigma} |A|^2 < \delta\, \epsilon\,, \tag{2.37}$$

then for all $0 < \sigma \leq r_0$ and $y \in B_{r_0 - \sigma}(x)$

$$\sigma^2\, |A|^2(y) \leq \delta\,. \tag{2.38}$$

PROOF: We shall give the proof for the case $M^n = \mathbb{R}^n$ and leave the necessary modifications for the general case to the reader, cf. Chapter 5.

Set $F = (r_0 - r)^2\, |A|^2$ on $B_{r_0} \cap \Sigma$ and observe that F vanishes on ∂B_{r_0}. Let $x_0 \in \Sigma$ be a point where F achieves its maximum and note that it is enough to show that $F(x_0) < \delta$. Suppose not, so that $F(x_0) \geq \delta$, and let $\sigma > 0$ with $2\,\sigma < r_0 - r(x_0)$ be such that

$$4\,\sigma^2\, |A|^2(x_0) = \delta\,. \tag{2.39}$$

Since F achieves its maximum at x_0,

$$\begin{aligned} \sup_{B_\sigma(x_0)\cap\Sigma} \sigma^2\, |A|^2 &= \sup_{B_\sigma(x_0)\cap\Sigma} \sigma^2\, \frac{F}{(r_0 - r)^2} \\ &\leq \frac{4\,\sigma^2}{(r_0 - r(x_0))^2} \sup_{B_\sigma(x_0)\cap\Sigma} F \leq \frac{4\,\sigma^2}{(r_0 - r(x_0))^2}\, F(x_0) \\ &\leq 4\,\sigma^2\, |A|^2(x_0) = \delta\,. \end{aligned} \tag{2.40}$$

From this, we conclude that

$$|A|^2(x_0) = \frac{\delta\,\sigma^{-2}}{4} \tag{2.41}$$

and

$$\sup_{B_\sigma(x_0)\cap\Sigma} |A|^2 \leq \delta\,\sigma^{-2}\,. \tag{2.42}$$

After rescaling the ball $B_\sigma(x_0)$ to unit size, we have

$$|A|^2(x_0) = \frac{\delta}{4} \tag{2.43}$$

and

$$\sup_{B_1(x_0)\cap\Sigma} |A|^2 \leq \delta \leq 1\,. \tag{2.44}$$

By Simons' inequality on $B_1(x_0) \cap \Sigma$,

$$\Delta |A|^2 \geq -2\,|A|^2\,. \tag{2.45}$$

The desired contradiction now follows from the mean value inequality. Namely, Corollary 1.12 implies that

$$\frac{\delta}{4} = |A|^2(x_0) \leq \mathrm{e}\, \frac{\int_{B_1(x_0)} |A|^2}{\pi} < \frac{\mathrm{e}}{\pi}\, \delta\, \epsilon\,, \tag{2.46}$$

which is a contradiction provided that ϵ is chosen sufficiently small. ∎

Before going on, we will make some remarks about this theorem. For this, it will convenient to let $B_s^\Sigma(x_0)$ denote the intrinsic ball of radius s centered at $x_0 \in \Sigma$. We will use this notation for the remainder of this section.

First, Theorem 2.3 holds with intrinsic balls instead of extrinsic balls (with slightly different constants). To see this, let r denote the intrinsic distance instead, replace extrinsic balls with intrinsic balls, and follow the argument verbatim up to (2.45). At this point, arguing as in Lemma 2.2, (2.44) implies that $B_1^\Sigma(x_0)$ contains the connected component of $B_{1/2}(x_0)$ which contains x_0. We can now replace $B_1(x_0)$ with $B_{1/2}(x_0)$ and apply the mean value inequality to complete the proof as before.

Second, by Lemma 2.2, the curvature bounds of the previous result, (2.38) imply that the connected components of $B_{\sigma/4}(y) \cap \Sigma$ are graphical with bounded gradient and Hessian. This implies that the associated minimal surface equation is a uniformly elliptic divergence form equation with C^1 coefficients. Standard elliptic theory then yields uniform estimates (see, for instance, corollary 16.7 of [GiTr]). This will be used later in Chapter 5 when we study compactness theorems for minimal surfaces.

Theorem 2.4 (E. Heinz [He]) *Let D_{r_0} be a disk in $\mathbb{R}^2$ of radius r_0. Suppose that $u : D_{r_0} \to \mathbb{R}$ satisfies the minimal surface equation; then for $\Sigma = \mathrm{Graph}_u$ and $0 < \sigma \leq r_0$*

$$\sigma^2 \sup_{D_{r_0-\sigma}} |A|^2 \leq C\,. \tag{2.47}$$

PROOF: Clearly, the theorem would follow provided we can show the case where $\sigma = r_0$. We have already shown in Corollary 1.15 of the previous chapter that, for any $\kappa > 1$,

$$\int_{B_{r_0/\sqrt{\kappa}} \cap \mathrm{Graph}_u} |A|^2 \leq \frac{16\, C^2\ \mathrm{e}^2\ \mathrm{Area}(\mathbf{S}^2)}{\log \kappa}\,. \tag{2.48}$$

We choose $\kappa > 1$ so that

$$\frac{16\ \mathrm{e}^2\ C^2\ \mathrm{Area}(\mathbf{S}^2)}{\log \kappa} < \epsilon\,, \tag{2.49}$$

where ϵ is given by Theorem 2.3 and the theorem now easily follows. ∎

So far we have seen that both minimal surfaces with small total curvature and minimal graphs satisfy a priori curvature estimates. We will see below that small area also implies a curvature estimate (see Theorem 2.10). In fact, for simply connected embedded minimal surfaces, any bound at all on the area or total curvature implies a curvature estimate. This argument applies more generally to surfaces with quasi-conformal Gauss maps (cf. (5.6)), but we will state only the following special case:

Theorem 2.5 (Schoen-Simon [ScSi]) *Let* $0 \in \Sigma^2 \subset B_{r_0} = B_{r_0}(0) \subset \mathbb{R}^3$ *be an embedded simply connected minimal surface with* $\partial \Sigma \subset \partial B_{r_0}$. *If* $\mu > 0$ *and either*

$$\operatorname{Area}(\Sigma) \leq \mu \, r_0^2 \, , \text{ or} \tag{2.50}$$

$$\int_\Sigma |A|^2 \leq \mu \, , \tag{2.51}$$

then for the connected component Σ' *of* $B_{r_0/2} \cap \Sigma$ *with* $0 \in \Sigma'$ *we have*

$$\sup_{\Sigma'} |A|^2 \leq C \, r_0^{-2} \tag{2.52}$$

for some $C = C(\mu)$.

As an immediate consequence, letting $r_0 \to \infty$ gives Bernstein-type theorems for embedded simply connected minimal surfaces with either bounded density or finite total curvature. Note that Enneper's surface shows that embeddedness is essential.

For stable minimal surfaces in three-manifolds, no assumption on the total curvature, area, or topology is needed to get curvature estimates. This is not known in higher dimensions; we will discuss the higher dimensional case in the next two sections.

The key fact in two dimensions is that stability itself implies a priori bounds on the geometry of the surface. The first local result in this direction was given by Schoen in [Sc] where he estimated the conformal factor of a stable minimal immersion (with trivial normal bundle) into a manifold with Ricci curvature bounded below.

The following result gives local area and total curvature estimates for intrinsic balls in a stable minimal surface assuming only a lower bound on the scalar curvature:

Theorem 2.6 (Colding-Minicozzi [CM11]) *Let* Σ^2 *be an immersed stable minimal surface in* M^3 *with trivial normal bundle,* $1 \geq 6 \, r_0^2 \, \Lambda \geq 0$ *and* $\operatorname{Scal}_M \geq -3\Lambda$. *If* $B^\Sigma_{r_0} = B^\Sigma_{r_0}(x) \subset \Sigma$ *with* $B^\Sigma_{r_0} \cap \partial \Sigma = \emptyset$, *then*

$$r_0^{-2} \, \operatorname{Area}(B^\Sigma_{r_0}) + \int_{B^\Sigma_{r_0}} \left[|A|^2 + 2 \left(\operatorname{Scal}_M + 3\Lambda \right) \right] (1 - r/r_0)^2 \leq 4 \, \pi \, . \tag{2.53}$$

If $\operatorname{Ric}_M \geq -\Lambda$, *then for any integer* $m > 0$

$$\int_{B^\Sigma_{\mathrm{e}^{-2\,m}\, r_0}} |A|^2 + \left(\operatorname{Ric}_M (N, N) + \Lambda \right) \leq 4 \, \pi \, \Lambda \, r_0^2 \, \mathrm{e}^{-2m} + 4 \, \pi \, \mathrm{e}^2 / m \, . \tag{2.54}$$

As immediate consequences, if the scalar or Ricci curvature is nonnegative (i.e., $\Lambda = 0$), then we can let $r_0 \to \infty$ in Theorem 2.6 to obtain Bernstein-type theorems. First, if $\mathrm{Ric}_M \geq 0$, then Σ must be parabolic and totally geodesic (this is the Bernstein theorem of [FiSc]). Second, if $\mathrm{Scal}_M \geq 0$, then Σ must be parabolic and have finite total second fundamental form (the first part of this is proven in [FiSc]; the second part is proven in [CM11]). See [CM11] for further discussion in this direction, as well as extensions to manifolds with negative scalar and Ricci curvature.

The following is now a consequence of (2.54) and Theorem 2.3:

Theorem 2.7 (R. Schoen [Sc]) *If $\Sigma^2 \subset M^3$ is an immersed stable minimal surface with trivial normal bundle and $B^{\Sigma}_{r_0} = B^{\Sigma}_{r_0}(x) \subset \Sigma \setminus \partial\Sigma$, where $|K_M| \leq k^2$ and $r_0 < \rho_1(\pi/k, k)$, then for some $C = C(k)$ and all $0 < \sigma \leq r_0$,*

$$\sup_{B^{\Sigma}_{r_0 - \sigma}} |A|^2 \leq C\,\sigma^{-2}\,. \tag{2.55}$$

Since intrinsic balls are contained in extrinsic balls:

Corollary 2.8 (R. Schoen [Sc]) *If $\Sigma^2 \subset B_{r_0} = B_{r_0}(x) \subset M^3$ is an immersed stable minimal surface with trivial normal bundle, where $|K_M| \leq k^2$, $r_0 < \rho_1(\pi/k, k)$, and $\partial\Sigma \subset \partial B_{r_0}$, then for some $C = C(k)$ and all $0 < \sigma \leq r_0$,*

$$\sup_{B_{r_0 - \sigma}} |A|^2 \leq C\,\sigma^{-2}\,. \tag{2.56}$$

See [CM11] for a generalization, and proof, of Theorem 2.7. Before going on, we will briefly discuss this generalization.

Let M^3 be a Riemannian 3-manifold. Given a function $\phi \geq 1$ on the unit sphere bundle of M, we can define a functional Φ on an immersed oriented surface Σ in M by

$$\Phi(\Sigma) = \int_{x \in \Sigma} \phi(x, N(x))\, dx\,. \tag{2.57}$$

The restriction to $\phi \geq 1$ is a convenient normalization. By analogy with the area functional (i.e., where $\phi \equiv 1$), a surface is said to be Φ-*stationary* if it is a critical point for Φ. A Φ-*stationary* surface is Φ-*stable* if its second variation is nonnegative for deformations of compact support. See [Al2] for more on the first and second variation of a parametric integrand. Φ is *elliptic* if there is some $\lambda > 0$ such that, for each $x \in M$, $v \to [\phi(x, v/|v|) - \lambda]|v|$ is a convex function of $v \in T_x M$.

Curvature estimates continue to hold for Φ-stable surfaces. Namely, let $D_1\phi$ and $D_2\phi$ denote the derivatives of ϕ with respect to the first and second components, respectively; then we have the following:

Theorem 2.9 (Colding-Minicozzi [CM11]) *There exist $\epsilon, \rho > 0$, and C, where ρ, C depend on $|\phi|_{C^{2,\alpha}}$, $|D_2\phi|_{C^{2,\alpha}}$, so that if $r_0 \leq \rho$, $|D_2\phi| + |D_2^2\phi| < \epsilon$,* *and* $B^\Sigma_{r_0} = B^\Sigma_{r_0}(x)$ *is a ball in an immersed oriented* Φ*-stable surface* $\Sigma \subset \mathbb{R}^3$ *with* $B^\Sigma_{r_0} \cap \partial\Sigma = \emptyset$*, then for all* $0 < \sigma \leq r_0$*,*

$$\sup_{B^\Sigma_{r_0-\sigma}} |A|^2 \leq C\,\sigma^{-2}\,. \tag{2.58}$$

As for the minimal surface equation, standard elliptic theory then implies higher derivative estimates (see [Si2]; cf. chapter 16 of [GiTr]).

We note that many of the standard tools for minimal surfaces do not hold for general Φ. In particular, monotonicity of area (i.e., Proposition 1.8) and Simons' inequality (i.e., Lemma 2.1) no longer hold. This introduces significant complications. For instance, Simons' inequality and an iteration argument were used in [Sc] to estimate the conformal factor for a stable immersion.

Finally, we will close this section by proving curvature estimates for (smooth) minimal surfaces with small *excess* following [CM8]. Allard's regularity theorem [Al1] shows that this estimate holds more generally without assuming smoothness.

Let $\Sigma^2 \subset B_1$ be a smooth minimal surface with $\partial\Sigma \subset \partial B_1$. Given $0 < s < 1$ and $x \in B_{1-s} \cap \Sigma$, the *excess* for Σ in $B_s(x)$ is defined to be

$$\Theta_x(s) - 1 = \frac{\text{Area}(B_s(x) \cap \Sigma)}{\pi\, s^2} - 1\,. \tag{2.59}$$

Monotonicity (i.e., Proposition 1.8) implies that the excess is nonnegative and monotone increasing in s.

Theorem 2.10 *There exist* $\epsilon > 0$ *and* $C < \infty$ *such that if* $0 \in \Sigma^2 \subset B_{2r_0}$ *is a smooth compact minimal surface with* $\partial\Sigma \subset \partial B_{2r_0}$ *and for all* $y \in B_{r_0} \cap \Sigma$

$$\Theta_y(r_0) - 1 < \epsilon\,, \tag{2.60}$$

then for all $0 < \sigma \leq r_0$ *and* $y \in B_{r_0-\sigma}(x)$

$$\sigma^2\, |A|^2(y) \leq 1\,. \tag{2.61}$$

PROOF: First, note that, by monotonicity, (2.60) implies that for all $s < r_0$ and $y \in B_{r_0} \cap \Sigma$

$$\Theta_y(s) - 1 < \epsilon\,. \tag{2.62}$$

Defining $F = (r_0 - r)^2\, |A|^2$ on $B_{r_0} \cap \Sigma$, it suffices to show that $F \leq 1$. Suppose not and argue as in the proof of Theorem 2.3, so that after rescaling Σ we have

$$|A|^2(x_0) = \frac{1}{4} \tag{2.63}$$

and

$$\sup_{B_1(x_0)\cap\Sigma} |A|^2 \leq 1\,. \tag{2.64}$$

Furthermore, (2.62) implies that for any $s < 1$

$$\text{Area}(B_s(x_0) \cap \Sigma) \leq \pi(1+\epsilon)s^2 \,. \tag{2.65}$$

To complete the proof, we will show that (2.63), (2.64), and (2.65) lead to a contradiction for $\epsilon > 0$ sufficiently small.

First, applying Theorem 2.3, (2.63) implies that

$$\pi\, C \leq \int_{B_{\frac{1}{32}}(x_0)\cap\Sigma} |A|^2 \,, \tag{2.66}$$

where C comes from Theorem 2.3. By Lemma 2.2, $B_{1/8}(x_0) \cap \Sigma$ is a graph with gradient bounded by 1. In particular, it is simply connected and for $x, y \in B_{1/8}(x_0) \cap \Sigma$

$$|x-y| \leq \text{dist}_\Sigma(x,y) \leq \sqrt{2}\,|x-y| \,. \tag{2.67}$$

In the remainder of this proof, B_s^Σ will denote the intrinsic ball in Σ of radius s centered at x_0. Combining (2.66) and (2.67), we get

$$\pi\, C \leq \int_{B^\Sigma_{1/16}} |A|^2 \,. \tag{2.68}$$

Since Σ is minimal, the Gauss equation implies that $K = K_\Sigma = -2|A|^2 \leq 0$. Letting $L(t)$ denote the length of ∂B_t^Σ, the usual comparison theorem for nonpositive curvature implies that

$$L(t) \geq 2\,\pi\, t \,. \tag{2.69}$$

The first variation formula for arc length gives

$$L'(t) = \int_{\partial B_t^\Sigma} k_g \,, \tag{2.70}$$

where k_g is the geodesic curvature. On the other hand, for $\frac{1}{16} \leq t \leq \frac{1}{8}$ we can estimate this using the Gauss-Bonnet theorem to get

$$L'(t) \geq 2\,\pi - \int_{B_t^\Sigma} K = 2\,\pi + 2\int_{B_t^\Sigma} |A|^2 \geq 2\,\pi(1+C) \,, \tag{2.71}$$

where the second inequality follows from (2.68). Using (2.69), integrating (2.71) yields for $\frac{3}{32} \leq t \leq \frac{1}{8}$

$$L(t) \geq \left(2 + \frac{C}{2}\right)\pi\, t \,. \tag{2.72}$$

By the coarea formula (i.e., (1.40)), $\text{Area}(B_t^\Sigma) = \int_0^t L(t)$. Therefore, using (2.69) for $0 \leq t < \frac{3}{32}$ and (2.72) for $\frac{3}{32} \leq t \leq \frac{1}{8}$, we have

$$\text{Area}(B^\Sigma_{\frac{1}{8}}) \geq \pi\left(1 + \frac{C}{32}\right)\frac{1}{64} \,. \tag{2.73}$$

Finally, (2.67) and (2.65) give

$$\text{Area}(B^{\Sigma}_{\frac{1}{8}}) \leq \text{Area}(B_{\frac{1}{8}}(x_0) \cap \Sigma) \leq \frac{\pi(1+\epsilon)}{64}\,, \tag{2.74}$$

which contradicts (2.73) for ϵ sufficiently small. This contradiction completes the proof. ∎

The above proof shows that, for surfaces, Theorem 2.3 implies the smooth version of Allard's estimate. In fact, these estimates can be shown to be equivalent by a similar argument using Proposition 5.16. This, and related results, are discussed in more detail in [CM8].

2.3 L^p Bounds of $|A|^2$ for Stable Hypersurfaces

In this section, we combine Simons' inequality with the stability inequality to show higher L^p bounds for the square of the norm of the second fundamental form for stable minimal hypersurfaces. In the next section, we will use this bound together with Simons' inequality to show curvature estimates for stable minimal hypersurfaces.

Theorem 2.11 (Schoen-Simon-Yau [ScSiYa]) *Suppose that $\Sigma^{n-1} \subset \mathbb{R}^n$ is an orientable stable minimal hypersurface. For all $p \in [2, 2+\sqrt{2/(n-1)})$ and each nonnegative Lipschitz function ϕ with compact support*

$$\int_\Sigma |A|^{2p}\,\phi^{2p} \leq C(n,p) \int_\Sigma |\nabla \phi|^{2p}\,. \tag{2.75}$$

PROOF: If we insert $\eta = |A|^{1+q}\,f$ in the stability inequality (i.e., Lemma 1.20) for $0 \leq q < \sqrt{2/(n-1)}$, we get

$$\begin{aligned} \int |A|^{4+2q}\,f^2 &\leq \int |f\,\nabla |A|^{1+q} + |A|^{1+q}\,\nabla f|^2 \\ &= (1+q)^2 \int f^2 |\nabla |A||^2\,|A|^{2q} + \int |A|^{2+2q} |\nabla f|^2 \\ &\quad + 2(1+q) \int f |A|^{1+2q} \langle \nabla f, \nabla |A| \rangle\,. \end{aligned} \tag{2.76}$$

Multiply Simons' inequality, that is,

$$|A|\,\Delta |A| + |A|^4 \geq \frac{2}{n-1} |\nabla |A||^2\,, \tag{2.77}$$

by $|A|^{2q} f^2$ and integrate to get

$$\begin{aligned} \frac{2}{n-1} \int |\nabla |A||^2\,|A|^{2q} f^2 &\leq \int |A|^{4+2q} f^2 + \int f^2 |A|^{1+2q} \Delta |A| \\ &= \int |A|^{4+2q} f^2 - 2 \int f |A|^{1+2q} \langle \nabla f, \nabla |A| \rangle \\ &\quad - (1+2q) \int f^2 |A|^{2q} |\nabla |A||^2\,. \end{aligned} \tag{2.78}$$

Combining (2.76) and (2.78) gives

$$\left(\frac{2}{n-1} - q^2\right) \int |A|^{2q} \, |\nabla |A||^2 \, f^2 \leq \int |A|^{2+2q} \, |\nabla f|^2 + 2q \int f |A|^{1+2q} \, \langle \nabla f, \, \nabla |A| \rangle \, . \tag{2.79}$$

Therefore, by the absorbing inequality, i.e, $xy \leq \epsilon \, x^2/2 + y^2/(2\epsilon)$, which holds for all $\epsilon > 0$, we get

$$\left(\frac{2}{n-1} - q^2\right) \int |A|^{2q} \, |\nabla |A||^2 \, f^2 \leq \int |A|^{2+2q} \, |\nabla f|^2 + \epsilon \, q \int f^2 \, |A|^{2q} \, |\nabla |A||^2 + \frac{q}{\epsilon} \int |\nabla f|^2 \, |A|^{2+2q} \, . \tag{2.80}$$

Hence,

$$\left(\frac{2}{n-1} - q^2 - \epsilon \, q\right) \int f^2 |A|^{2q} |\nabla |A||^2 \leq \left(1 + \frac{q}{\epsilon}\right) \int |\nabla f|^2 |A|^{2+2q} \, , \tag{2.81}$$

and therefore (for any $\epsilon < \frac{2/(n-1)-q^2}{q}$)

$$\begin{aligned} \int |A|^{4+2q} f^2 &\leq 2(1+q)^2 \int f^2 |A|^{2q} |\nabla |A||^2 + 2 \int |A|^{2q+2} |\nabla f|^2 \\ &\leq \left(\frac{2(1+q)^2(1+q/\epsilon)}{\frac{2}{n-1} - q^2 - \epsilon \, q} + 2 \right) \int |A|^{2+2q} |\nabla f|^2 \, . \end{aligned} \tag{2.82}$$

If we set $p = 2 + q$ and $f = \phi^p$, then $2 \leq p < 2 + \sqrt{2/(n-1)}$ and for some $c = c(n,p)$

$$\int |A|^{2p} \phi^{2p} \leq c \int |A|^{2p-2} \phi^{2p-2} |\nabla \phi|^2 \, . \tag{2.83}$$

Finally, if we apply Young's inequality (i.e., $xy \leq x^a/a + y^b/b$ for $1/a + 1/b = 1$) to $x = \delta \, |A|^{2p-2} \phi^{2p-2}$, $y = |\nabla \phi|^2/\delta$, $a = \frac{p}{p-1}$, and $b = p$, then

$$\int |A|^{2p} \phi^{2p} \leq c \, \frac{p-1}{p} \delta^{\frac{p}{p-1}} \int |A|^{2p} \phi^{2p} + \frac{c}{p} \, \delta^{-p} \int |\nabla \phi|^{2p} \, . \tag{2.84}$$

The claim easily follows from this if we choose $\delta > 0$ sufficiently small. ∎

2.4 Bernstein Theorems and Curvature Estimates

In this section, we will first show a generalization of the Bernstein theorem proven in Chapter 1.

Theorem 2.12 (S. Bernstein [Be] for $n = 3$, E. De Giorgi [DG] for $n = 4$, F. J. Almgren, Jr. [Am1] for $n = 5$, and J. Simons [Sim] for $n \leq 8$) *If $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is an entire solution to the minimal surface equation and $n \leq 8$, then u is a linear function.*

PROOF: (The proof we give follows [ScSiYa]). We will prove this result here only for $n \leq 6$. Suppose that $n \leq 6$ and fix $x \in \mathbb{R}^n$. For each $r > 0$ let ϕ be the cutoff function with $\phi|B_r \equiv 1$, $\phi|B_{2r} \equiv 0$ and such that ϕ decays linearly in the radial direction on the annulus $B_{2r} \setminus B_r$. By combining the L^p bound of $|A|^2$ from Theorem 2.11 for this cutoff function and $2p = 4 + \sqrt{7/5} < 4 + \sqrt{8/(n-1)}$ with the volume bound for minimal graphs (1.20), we get

$$\int_{B_r \cap \Sigma} |A|^{4+\sqrt{7/5}} \leq C(n,p)\, r^{-4-\sqrt{7/5}} \ \mathrm{Vol}(B_{2r} \cap \Sigma) \leq \bar{C}\, r^{n-5-\sqrt{7/5}} \,. \tag{2.85}$$

Since $n - 5 - \sqrt{7/5} < 0$, by letting r go to infinity we get that $|A|^2 \equiv 0$. The claim now easily follows. ∎

In fact, we get curvature estimates for minimal graphs. Namely, we have the following:

Theorem 2.13 (E. Heinz [He] for $n = 3$, Schoen-Simon-Yau [ScSiYa] for $n \leq 6$, and L. Simon [Si1] for $n \leq 8$) *If $u : D_{r_0} \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is a solution to the minimal surface equation on the $(n-1)$-dimensional disk of radius r_0 and $n \leq 8$, then for $0 < \sigma \leq r_0$*

$$\sup_{D_{r_0 - \sigma}} |A|^2 \leq C(n)\, \sigma^{-2} \,. \tag{2.86}$$

PROOF: (Again, we prove this only for $n \leq 6$). The proof closely follows that of Theorem 2.4. Namely, taking $2p = n$ in Theorem 2.11 and letting ϕ be a logarithmic cutoff function (as in the proof of Corollary 1.15), we get that the area bounds for minimal graphs imply that

$$\int_{B_{\sqrt{\kappa}\, R} \cap \mathrm{Graph}_u} |A|^n \leq \frac{C_1}{\log \kappa} \,, \tag{2.87}$$

where $C_1 = C_1(n)$, $\kappa > 1$, and $\kappa\, R \leq r_0$. On the other hand, Simons' inequality implies that

$$\Delta |A|^n \geq -C_2\, |A|^{n+2} \,, \tag{2.88}$$

so we can argue as in Theorem 2.3 to get a corresponding curvature estimate. ∎

2.5 Minimal Cones

We close this chapter with a discussion of minimal cones in Euclidean space. The study of these cones has been important both in the generalizations of the theorem of Bernstein and on issues of local regularity.

If N^{k-1} is a (smooth) submanifold of $\mathbf{S}^{n-1} \subset \mathbb{R}^n$, then the *cone* over N will be denoted by $C(N)$ and, as a set, $C(N) = \{x \in \mathbb{R}^n \mid x/|x| \in N\}$. Therefore $C(N)$ is a smooth k-dimensional submanifold away from the origin (which is the vertex of the cone). By definition, cones are invariant under dilations about the origin. First, we have the following simple lemma whose proof is left for the reader:

Lemma 2.14 *A submanifold $N^{k-1} \subset \mathbf{S}^{n-1}$ is minimal if and only if its Euclidean mean curvature is everywhere normal to $\mathbf{S}^{n-1} \subset \mathbb{R}^n$.*

Let $x = (x_1, \ldots, x_n)$ denote the position vector in $\mathbb{R}^n$ and set $r = |x|$. It follows from this lemma that if $N^{k-1} \subset \mathbf{S}^{n-1}$ is minimal, then the Euclidean mean curvature of N is given by the vector field $\Delta x = (\Delta x_1, \ldots, \Delta x_n)$, where Δ is the metric Laplacian on N.

We now get the following lemma:

Lemma 2.15 *If $N^{k-1} \subset \mathbf{S}^{n-1}$ is a minimal submanifold, then the coordinate functions are eigenfunctions with eigenvalue $k-1$.*

PROOF: Since N is minimal, Δx is normal to $\mathbf{S}^{n-1}$. Therefore, $\Delta x = x\, f$ for some function f. Since $x \in \mathbf{S}^{n-1}$, $|x|^2 = 1$ and hence

$$0 = \Delta |x|^2 = 2\,\langle x, \Delta x\rangle + 2|\nabla x|^2 = 2\,f + 2(k-1)\,. \tag{2.89}$$

∎

Example 2.16 If $\mathbf{S}^{k-1}$ is a totally geodesic $(k-1)$-sphere in $\mathbf{S}^{n-1}$, then $C(\mathbf{S}^{k-1})$ is a k-dimensional plane through the origin in $\mathbb{R}^n$.

Let $N^{k-1} \subset \mathbf{S}^{n-1}$ be an immersed submanifold (not necessarily minimal). Since

$$\nabla_{C(N)} u = r^{-1}\, \nabla_N u(r^{-1}\,\cdot) + \frac{\partial u}{\partial r}\,\frac{\partial}{\partial r}\,, \tag{2.90}$$

a direct computation shows that the Laplacians of N and $C(N)$ are related by the following simple formula at $x \neq 0$:

$$\Delta_{C(N)} u = r^{-2} \Delta_N u(r^{-1}\,x) + (k-1)\, r^{-1}\frac{\partial}{\partial r} u + \frac{\partial^2}{\partial r^2} u\,. \tag{2.91}$$

Lemma 2.17 *If $N^{k-1} \subset \mathbf{S}^{n-1}$ is a minimal submanifold, then $C(N) \subset \mathbb{R}^n$ is minimal.*

PROOF: It suffices to show that each coordinate function x_i is harmonic on $C(N)$. We write $x_i = r\, u_i$ where u_i is independent of r and so that x_i and u_i agree on $N \subset \mathbf{S}^{n-1}$. Lemma 2.15 and (2.91) give

$$\begin{aligned} \Delta_{C(N)} x_i &= r^{-1} \Delta_N u_i + u_i\,(k-1)\, r^{-1}\frac{\partial}{\partial r} r + u_i\,\frac{\partial^2}{\partial r^2} r \\ &= -(k-1)\, r^{-1} u_i + (k-1)\, r^{-1} u_i = 0\,. \end{aligned} \tag{2.92}$$

∎

Minimal cones arise in the generalizations of Bernstein's original theorem and in the study of local regularity for minimal submanifolds. The key fact here is that since cones are invariant under dilations, the ratio $\mathrm{Vol}(C(N) \cap B_r)/r^k$ is constant.

Conversely, if Σ^k is minimal and this ratio is constant, then we have equality in the monotonicity formula. It follows that $\nabla|x|^2$ is tangent to Σ almost everywhere and hence that Σ is invariant under dilations.

Suppose now that Σ^k is an area-minimizing minimal hypersurface. Arguing as we did in (1.20) for minimal graphs in Chapter 1, the density V_Σ is uniformly bounded. If $r_j \to 0$, then the sequence $\Sigma_j = r_j\,\Sigma$ of dilated surfaces is also area-minimizing. The bound on V_{Σ_j} implies the existence of a convergent subsequence (say, as area-minimizing currents or stationary varifolds); see the next chapter for more on this. The (not necessarily unique) limit is denoted by Σ_∞ and is also area-minimizing. It follows from the monotonicity formula that $\mathrm{V}_{\Sigma_\infty} = \mathrm{V}_\Sigma$ is also equal to the density of Σ_∞ at the origin. Therefore, we have equality in the monotonicity formula and consequently Σ_∞ is an area-minimizing minimal cone. This argument, which dates back to W. Fleming [Fl], allows one to prove the Bernstein theorem by showing the nonexistence of area-minimizing cones.

Theorem 2.18 (Fleming [Fl]) *If there is a nonlinear entire solution of the minimal surface equation on $\mathbb{R}^n$, then there is a singular area-minimizing cone in $\mathbb{R}^{n+1}$.*

Shortly after Fleming's result, E. De Giorgi showed in [DG] that the minimal cone given by Theorem 2.18 was in fact cylindrical (here cylindrical means that the cone, as a subset of $\mathbb{R}^n$, splits off a line isometrically, that is, $C(N) = \mathbb{R} \times C(N') \subset \mathbb{R}^n$). This implied that there actually existed a singular area-minimizing cone in $\mathbb{R}^n$.

The significance of this approach is that cones are much simpler to analyze. This connection is fundamental in the modern theory of regularity; an important illustration of this is the so-called dimension reduction argument of Federer. For instance, in dimension two such a cone must be a collection of planes through the origin; it is easy to see that this can only be minimizing when there is just a single plane. F. J. Almgren, Jr. showed that in dimension three such a cone was also a hyperplane. Finally, J. Simons proved the same theorem for $n \leq 7$.

Theorem 2.19 (F. J. Almgren, Jr. [Am1] for $n = 3$ and J. Simons [Sim] for $n \leq 7$) *The hyperplanes are the only stable minimal hypercones in $\mathbb{R}^n$ for $n \leq 7$.*

However, in 1969 Bombieri, De Giorgi, and Giusti [BDGG] gave an example of an area-minimizing singular cone in $\mathbb{R}^8$. In fact, they showed that for $m \geq 4$ the cones

$$(2.93) \quad C_m = \{(x_1, \ldots, x_{2m}) \mid x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2\} \subset \mathbb{R}^{2m}$$

are area-minimizing.

3

Weak Bernstein-Type Theorems

In this chapter, we will prove a generalization of the classical Bernstein theorem for minimal surfaces discussed in the previous chapters. The various Bernstein theorems imply that, through dimension seven, area-minimizing hypersurfaces must be affine. A weaker form of this is true in all dimensions by the Allard regularity theorem [Al1]. Namely, there exists $\delta = \delta(k, n) > 0$ such that if $\Sigma^k \subset \mathbb{R}^n$ is a k-dimensional complete immersed minimal submanifold with

$$\mathrm{Vol}(B_r(x) \cap \Sigma) \leq (1 + \delta)\ \mathrm{Vol}(B_r \subset \mathbb{R}^k) \tag{3.1}$$

for all x and r, then Σ is an affine k-plane. More generally, this theorem holds even when Σ has singularities, for instance, when Σ is a stationary integral k-varifold (see Definitions 3.3, 3.4, and 3.10 below).

In this chapter, we will show that, in fact, a bound on the density (see (3.31) for the definition of the density for a varifold) gives an upper bound for the dimension of the smallest affine subspace containing the minimal surface. We will deduce this theorem from the properties of the coordinate functions (in fact, more generally, from properties of harmonic functions) on k-rectifiable stationary varifolds of arbitrary codimension in Euclidean space. We refer to the original paper [CM4] for further discussion and more general results.

3.1 The Theory of Varifolds

We will begin by briefly describing the basic theory of varifolds. For convenience of notation, we will restrict ourselves to varifolds in $\mathbb{R}^n$; without much further work we could consider varifolds in smooth n-dimensional manifolds. Varifolds should be thought of as a generalization of submanifolds. This class of generalized submanifolds is broad enough to give general compactness results and allow for singularities but is narrow enough to still have geometric significance. For instance, we can define the first variation for a varifold.

In the theory of varifolds, surfaces are identified with certain Radon measures. The basic compactness result will then follow from the compactness theorem for Radon measures.

Definition 3.1 Let X be a locally compact separable space. A *Radon measure* μ is a Borel regular measure which is finite on compact subsets of X.

The following standard compactness result will be the main benefit of viewing submanifolds as Radon measures:

Theorem 3.2 (Compactness of Radon Measures) *Let X be a locally compact separable space and μ_j a sequence of Radon measures on X such that*

$$\sup_j \mu_j(U) < \infty \tag{3.2}$$

for any open set U with compact closure. Then there exists a Radon measure μ and a subsequence $\mu_{j'}$ such that $\mu_{j'} \to \mu$; that is, if f is a continuous function on X with compact support, then

$$\lim_{j' \to \infty} \int_X f \, d\mu_{j'} = \int_X f \, d\mu \, . \tag{3.3}$$

Let $\Sigma^k \subset \mathbb{R}^n$ be a smooth properly embedded submanifold. There is an obvious way to associate to Σ a Radon measure μ; namely, let

$$\mu(A) = \text{Vol}(A \cap \Sigma) \tag{3.4}$$

on a Borel set $A \subset \mathbb{R}^n$. Theorem 3.2 gives compactness for submanifolds (or the associated measures) as long as the submanifolds have uniform local area bounds. That is, suppose that $C < \infty$ and $\Sigma_j \subset \mathbb{R}^n$ is a sequence of k-dimensional submanifolds with

$$\text{Vol}(B_1(x) \cap \Sigma_j) < C \tag{3.5}$$

for all $x \in \mathbb{R}^n$ and all j. We define associated Radon measures μ_j as in (3.4). By Theorem 3.2, there exists a subsequence j' and a Radon measure μ so that the $\mu_{j'}$ converge weakly to μ.

The price that we have to pay for viewing a sequence of submanifolds as measures is that the limit is merely a measure (and need not exhibit any geometry). In particular, if we were attempting to extract an area-minimizing submanifold from a minimizing sequence, then we would not be able to say that the limit was minimal (i.e., had zero first variation).

Let $G(k,n)$ be the space of (unoriented) k-planes through the origin in $\mathbb{R}^n$ (so that $G(n-1,n)$ is projective $(n-1)$ space). Let $\pi : \mathbb{R}^n \times G(k,n) \to \mathbb{R}^n$ denote projection $\pi(x,\omega) = x$. Note that π is proper. We will also define the divergence and gradient with respect to a k-dimensional plane ω at $x \in \mathbb{R}^n$ as follows: If X is a C^1 vector field, then we define the divergence with respect to (x,ω) by

$$\text{div}_\omega X = \sum_{i=1}^k \langle E_i, \nabla_{E_i} X \rangle \, , \tag{3.6}$$

where E_i is an orthonormal basis for ω and ∇ is the Euclidean derivative. Likewise, if u is a C^1 function, then we set

$$\nabla_\omega u = \sum_{i=1}^{k} E_i(u)\, E_i\,. \tag{3.7}$$

Definition 3.3 (Varifold) A k-*varifold* T on $\mathbb{R}^n$ is a Radon measure on $\mathbb{R}^n \times G(k,n)$. The weight μ_T of T is the Radon measure on $\mathbb{R}^n$ given by $\mu_T(E) = T(\pi^{-1}E)$, the support of T is the support of μ_T, and the mass of T on a set $U \subset \mathbb{R}^n$ is just $\mu_T(U)$.

Once again, let $\Sigma^k \subset \mathbb{R}^n$ be a smooth properly embedded submanifold. Define $T\Sigma \subset \mathbb{R}^n \times G(k,n)$ by $(x,\omega) \in T\Sigma$ if ω is the tangent k-plane to Σ at x. Σ gives rise to a k-varifold T defined by

$$T(A) = \mathrm{Vol}(\pi(A \cap T\Sigma)) \tag{3.8}$$

on a Borel set $A \subset \mathbb{R}^n \times G(k,n)$. Here Vol is k-dimensional volume. In this case, the mass of T on a set U is simply the volume of Σ in U.

With this definition, it is easy to see how varifolds behave under mappings. Namely, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 mapping (so that F and dF are C^1), then F induces a C^1 map $\mathcal{F}$ from $\mathbb{R}^n \times G(k,n)$ to $\mathbb{R}^n \times \cup_{\ell \leq k} G(\ell, n)$:

$$\mathcal{F}(x,\omega) = (F(x), dF_x(\omega))\,. \tag{3.9}$$

If T is a k-varifold and the mapping F is proper on the support of T, then we define the (weighted) push-forward varifold $F(T)$ by

$$F(T)(A) = \int_{\mathcal{F}^{-1}A} J_\omega F(x)\, dT(x,\omega)\,, \tag{3.10}$$

where $J_\omega F(x)$ is the Jacobian factor given by

$$J_\omega F(x) = \left(\det(dF_x|_\omega)^t \circ (dF_x|_\omega)\right)^{\frac{1}{2}}\,. \tag{3.11}$$

The integrand (3.11) vanishes on the set where $\mathcal{F}^{-1}$ is not defined, and hence the push-forward varifold $F(T)(A)$ given by (3.10) is well-defined. When T comes from a smooth surface Σ, $J_\omega F(x)$ is just the square root of the determinant of the pullback metric and then (3.10) is just the usual area formula (see, for instance, [Fe]).

Using this transformation rule for varifolds, we can compute the first variation of "mass" for a varifold. Suppose that $F(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a one-parameter family of C^2 diffeomorphisms such that $F(\cdot, 0)$ is the identity and $F(\cdot, t)$ is the identity outside of a fixed compact set K. Let $F_t(x)$ denote the vector field $\frac{\partial F}{\partial t}(x,0)$ (so that F_t vanishes outside a compact set K) and suppose that $K \subset U \subset \mathbb{R}^n$. Differentiating under the integral sign in (3.10) and arguing precisely as in Chapter 1, we get

$$\frac{d}{dt}_{t=0} \mu_{F(T,t)}(U) = \int_{U \times G(k,n)} \mathrm{div}_\omega\, F_t(x)\, dT(x,\omega)\,. \tag{3.12}$$

We will say that a varifold is stationary if it has zero first variation for any compactly supported C^1 vector field.

Definition 3.4 A varifold T is *stationary* if for any compactly supported C^1 vector field X we have

$$\int_{\mathbb{R}^n \times G(k,n)} \operatorname{div}_\omega X \, dT(x,\omega) = 0 \,. \tag{3.13}$$

Note that, by (3.12), this definition is equivalent to requiring that the varifold is a critical point for the mass functional for all compactly supported C^2 variations. Observe also that if a varifold arises from a smooth submanifold, then the first variation formula implies that it is stationary if and only if it has zero mean curvature.

An important consequence of this definition is the following supplement to the compactness theorem for varifolds discussed earlier:

Proposition 3.5 *If T_j is a sequence of stationary varifolds which converges weakly to a varifold T, then T is also stationary.*

PROOF: Given a C^1 vector field X with compact support, we can define a compactly supported continuous function f by

$$f(x,\omega) = \operatorname{div}_\omega X \,. \tag{3.14}$$

Since the T_j are stationary,

$$\int_{\mathbb{R}^n \times G(k,n)} f(x,\omega) \, dT_j(x,\omega) = \int_{\mathbb{R}^n \times G(k,n)} \operatorname{div}_\omega X \, dT_j(x,\omega) = 0 \,. \tag{3.15}$$

Therefore by varifold convergence and by the compact support of f on $\mathbb{R}^n \times G(k,n)$, we get

$$\begin{aligned} \int_{\mathbb{R}^n \times G(k,n)} \operatorname{div}_\omega X \, dT(x,\omega) &= \int_{\mathbb{R}^n \times G(k,n)} f \, dT(x,\omega) \\ &= \lim_{j \to \infty} \int_{\mathbb{R}^n \times G(k,n)} f \, dT_j(x,\omega) = 0 \,. \end{aligned} \tag{3.16}$$

∎

Many of the earlier results about smooth minimal surfaces can be extended to stationary varifolds by inserting appropriate choices of vector fields in (3.13). We will give the arguments for two of these extensions below: the harmonicity of the coordinate functions and the monotonicity formula. These results should be compared with those of Chapter 1, and with Propositions 1.6, 1.8, and 1.11, in particular.

A function u on $\mathbb{R}^n$ is said to be *weakly harmonic* on a varifold T if, for any smooth function η with compact support,

$$\int_{\mathbb{R}^n \times G(k,n)} \langle \nabla_\omega \eta, \nabla_\omega u \rangle \, dT(x,\omega) = 0 \,. \tag{3.17}$$

Similarly we say that a function u on $\mathbb{R}^n$ is *weakly subharmonic* on a varifold T if, for any smooth nonnegative function η with compact support,

$$\int_{\mathbb{R}^n \times G(k,n)} \langle \nabla_\omega \eta, \nabla_\omega u \rangle \, dT(x,\omega) \leq 0 \, . \tag{3.18}$$

Proposition 3.6 (Harmonicity of the Coordinate Functions) *A varifold $T^k \subset \mathbb{R}^n$ is a stationary varifold if and only if the coordinate functions x_i are weakly harmonic.*

PROOF: Let η be a smooth function with compact support and set $e_i = \nabla x_i$. For any unit vector E_j we have $\nabla_{E_j}(\eta \, e_i) = (\nabla_{E_j} \eta) e_i$, and hence for any k-plane ω

$$\operatorname{div}_\omega(\eta \, e_i) = \langle \nabla_\omega \eta, e_i \rangle = \langle \nabla_\omega \eta, \nabla_\omega x_i \rangle \, , \tag{3.19}$$

where $\nabla_\omega \eta$ is the projection of $\nabla \eta$ to ω. Therefore,

$$\int_{\mathbb{R}^n \times G(k,n)} \operatorname{div}_\omega(\eta \, e_i) \, dT(x,\omega) = \int_{\mathbb{R}^n \times G(k,n)} \langle \nabla_\omega \eta, \nabla_\omega x_i \rangle \, dT(x,\omega) \, . \tag{3.20}$$

The claim easily follows from (3.20). ∎

Similarly, the monotonicity formula may be generalized to this setting. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a nonnegative function and set $r = |x|$. Given any vector E, we have

$$\nabla_E x = E \, , \tag{3.21}$$

where $x = (x_1, \ldots, x_n)$. For ω a k-plane with orthonormal basis E_i, we use (3.21) to compute

$$\begin{aligned} \operatorname{div}_\omega(\eta(r) x) &= \sum_{i=1}^{k} \langle E_i, \nabla_{E_i}(\eta(r) x) \rangle \\ &= k \, \eta(r) + \eta'(r) \, \langle \nabla_\omega r, x \rangle = k \, \eta(r) + r \, \eta'(r) \, |\nabla_\omega r|^2 \, . \end{aligned} \tag{3.22}$$

Let ω^N denote the orthogonal $(n-k)$-plane to ω. We have

$$1 = |\nabla r|^2 = |\nabla_\omega r|^2 + |\nabla_{\omega^N} r|^2 \, . \tag{3.23}$$

Proposition 3.7 (Monotonicity for Stationary Varifolds) *Suppose that $T^k \subset \mathbb{R}^n$ is a stationary varifold and $x_0 \in \mathbb{R}^n$; then for all $0 < s < t$*

$$\begin{aligned} & t^{-k} \, \mu_T(B_t(x_0)) - s^{-k} \, \mu_T(B_s(x_0)) \\ &\quad = \int_{(B_t(x_0) \setminus B_s(x_0)) \times G(k,n)} r^{-k} \, |\nabla_{\omega^N} r|^2 \, dT(x,\omega) \, . \end{aligned} \tag{3.24}$$

PROOF: After a translation, it suffices to assume that $x_0 = 0$. Let ϕ be a nonnegative cutoff function with $\phi'(s) \leq 0$ which is identically one on $[0, \frac{1}{2}]$ and supported on $[0,1]$. Fix s for the moment and let $\eta(r) = \phi(\frac{r}{s})$ so that

$$r\,\eta'(r) = -s\,\frac{d}{ds}\left(\phi\left(\frac{r}{s}\right)\right)\,. \tag{3.25}$$

Since T is stationary, integrating (3.22) and using (3.23) gives

$$\begin{aligned} 0 &= \int \operatorname{div}_\omega(\eta(r)x)\,dT(x,\omega) \\ &= \int \left(k\,\eta(r) + r\,\eta'(r)\right)\,dT(x,\omega) - \int r\,\eta'(r)\,|\nabla_{\omega^N} r|^2\,dT(x,\omega)\,. \end{aligned} \tag{3.26}$$

Substituting (3.25) into (3.26) gives

$$\begin{aligned} &\int k\,\phi\left(\frac{r}{s}\right) - s\,\frac{d}{ds}\left(\phi\left(\frac{r}{s}\right)\right)\,dT(x,\omega) \\ &\qquad = -s\int \frac{d}{ds}\left(\phi\left(\frac{r}{s}\right)\right)\,|\nabla_{\omega^N} r|^2\,dT(x,\omega)\,. \end{aligned} \tag{3.27}$$

Multiplying through by s^{-k-1}, we may rewrite (3.27) as

$$\begin{aligned} &\frac{d}{ds}\left(s^{-k}\int \phi\left(\frac{r}{s}\right)\,dT(x,\omega)\right) \\ &\qquad = s^{-k}\frac{d}{ds}\left(\int \phi\left(\frac{r}{s}\right)\,|\nabla_{\omega^N} r|^2\,dT(x,\omega)\right)\,. \end{aligned} \tag{3.28}$$

If we let ϕ increase to the characteristic function of $[0,1]$ and apply the monotone convergence theorem to (3.28), we get

$$\begin{aligned} \frac{d}{ds}\left(s^{-k}\,\mu_T(B_s)\right) &= s^{-k}\frac{d}{ds}\left(\int_{B_s\times G(k,n)} |\nabla_{\omega^N} r|^2\,dT(x,\omega)\right) \\ &= \frac{d}{ds}\left(\int_{B_s\times G(k,n)} r^{-k}\,|\nabla_{\omega^N} r|^2\,dT(x,\omega)\right)\,. \end{aligned} \tag{3.29}$$

Equation (3.29) holds both in the sense of distributions and for almost every s. To see this, we use the monotonicity of both $\mu_T(B_s)$ and the integral on the right-hand side of (3.29). Integrating (3.29) from s to t yields

$$\begin{aligned} &t^{-k}\,\mu_T(B_t) - s^{-k}\,\mu_T(B_s) \\ &\qquad = \int_{(B_t\setminus B_s)\times G(k,n)} r^{-k}\,|\nabla_{\omega^N} r|^2\,dT(x,\omega)\,. \end{aligned} \tag{3.30}$$

∎

In Chapter 1, we defined the density for a minimal submanifold. We can now extend this to stationary varifolds. Namely, given a stationary k-varifold $T^k \subset \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, and $s > 0$, we define the density by

$$\Theta_{x_0}(s) = \frac{\mu_T(B_s(x_0))}{\mathrm{Vol}(B_1 \subset \mathbb{R}^k)\, s^k} \,. \tag{3.31}$$

A simple modification of the argument in Proposition 1.8 implies that $\Theta_{x_0}(s)$ is a nondecreasing function of s.

Similarly, the more general mean value inequality also extends to stationary varifolds. The necessary modifications will be left to the reader.

Proposition 3.8 (The Mean Value Inequality for Stationary Varifolds) *If $T \subset \mathbb{R}^n$ is a stationary k-varifold, $x_0 \in \mathbb{R}^n$, f is a nonnegative weakly subharmonic function on T, and $0 < s < t$, then*

$$t^{-k} \int_{B_t(x_0)} f \, d\mu_T \geq s^{-k} \int_{B_s(x_0)} f \, d\mu_T \,. \tag{3.32}$$

Thus far we have been discussing the theory of general varifolds. The varifolds which come from smooth submanifolds have a good deal of additional structure. In particular, the associated measures are supported on smooth submanifolds which have tangent spaces at every point. The class of varifolds known as rectifiable varifolds have similar additional structure.

Let $\mathcal{H}^k$ denote k-dimensional Hausdorff measure.

Definition 3.9 A set $S \subset \mathbb{R}^n$ is said to be *k-rectifiable* if $S \subset S_0 \cup S_1$, where $\mathcal{H}^k(S_0) = 0$ and S_1 is the image of $\mathbb{R}^k$ under a Lipschitz map.

More generally, S is said to be *countably k-rectifiable* if $S \subset \cup_{\ell \geq 0} S_\ell$, where $\mathcal{H}^k(S_0) = 0$ and for $\ell \geq 1$ each S_ℓ is the image of $\mathbb{R}^k$ under a Lipschitz map.

One advantage of working with rectifiable sets is that they have tangent spaces at almost every point by Rademacher's theorem [Fe]. Consequently, we may associate a varifold to a rectifiable set, just as we did for smooth submanifolds. Rectifiable varifolds are then varifolds which are supported on rectifiable sets.

Definition 3.10 (Rectifiable Varifold) Let S be a countably k-rectifiable subset of $\mathbb{R}^n$ with $\mathcal{H}^k(S) < \infty$ and let θ be a positive locally $\mathcal{H}^k$ integrable function on S. Set T equal to the varifold associated to the set S (exactly as if S were a smooth submanifold). The associated varifold $T' = \theta\, T$ is called a *rectifiable varifold*. If θ is integer-valued, then T' is an *integral varifold*.

Henceforth, $\Sigma^k \subset \mathbb{R}^n$ will be a stationary rectifiable k-varifold. Associated to each such Σ is a Radon measure μ_Σ; abusing notation slightly, we will also use Σ to denote the (Hausdorff k-dimensional rectifiable) set on which the measure is supported. We shall make the standard assumption that the density is at least 1 on the support. Note that this class of generalized minimal submanifolds includes the case of embedded minimal submanifolds equipped with the intrinsic Riemannian metric.

3.2 The Weak Bernstein-Type Theorem

We will refer to the following result as the weak Bernstein-type theorem.

Theorem 3.11 (Colding-Minicozzi [CM4]) *If $\Sigma^k \subset \mathbb{R}^n$ is a k-rectifiable stationary varifold with (volume) density at least 1 almost everywhere and bounded from above by V_Σ, then Σ must be contained in some affine subspace of dimension at most $(k+1)\,\frac{k}{k-1}\;\mathrm{e}^8\;2^{k+4}\,\mathrm{V}_\Sigma$.*

The constant in Theorem 3.11 is not sharp, and is not the best constant which can be obtained from the arguments of [CM4]. However, the geometric dependence is sharp (namely, the bound is linear in V_Σ which is optimal).

Notice that Theorem 3.11 bounds the number of linearly independent coordinate functions on Σ in terms of its volume. When V_Σ is sufficiently small, Allard's theorem implies that Σ^k is planar and hence that there are only k independent coordinate functions. We note that it follows from Theorem 3.11 that the δ in the Allard regularity theorem is independent of n.

A calibration argument (see [Fe] or [Mr]), cf. (1.15) and (1.16), shows that complex submanifolds of $\mathbf{C}^n = \mathbb{R}^{2n}$ are absolutely area-minimizing (and hence minimal). Using this, it is easy to see that affine algebraic varieties are examples of stationary rectifiable varifolds with bounded density.

We saw earlier that the coordinate functions are weakly harmonic on Σ (Proposition 3.6; cf. Proposition 1.6). Theorem 3.11 will follow from a bound for the dimensions of certain spaces of harmonic functions on Σ, namely the so-called harmonic functions of polynomial growth.

Definition 3.12 (Harmonic Functions of Polynomial Growth) We will define $\mathcal{H}_d(\Sigma)$ to be the linear space of harmonic functions with polynomial growth of order at most d. That is, $u \in \mathcal{H}_d(\Sigma)$ if u is harmonic and there exists some $C < \infty$ so that $|u(x)| \leq C(1 + |x|^d)$.

With this definition, the coordinate functions x_i are in $\mathcal{H}_1(\Sigma)$. For example, on the catenoid $\Sigma_c^2 \subset \mathbb{R}^3$ centered on the x_3-axis (see Example 1.4), the function x_3 grows slower than any power of the intrinsic distance; however, it is not in $\mathcal{H}_d(\Sigma_c^2, \mathbb{R})$ for any $d < 1$.

Theorem 3.13 (Colding-Minicozzi [CM4]) *Let Σ^k be a stationary k-rectifiable varifold with density at least 1 almost everywhere and bounded from above by $\mathrm{V}_\Sigma < \infty$. For any $d \geq 1$,*

$$\dim \mathcal{H}_d(\Sigma) \leq C\;\mathrm{V}_\Sigma\;d^{k-1}\,, \tag{3.33}$$

where $C = (k+1)\,\frac{k}{k-1}\;\mathrm{e}^8\;2^{k+4}$.

The proofs of the finite dimensionality results here consist of the following two independent steps. First, we will reduce the problem to bounding the number of $L^2(B_r \cap \Sigma)$-orthonormal harmonic functions on Σ with a uniform bound on the

$L^2(B_{\Omega r} \cap \Sigma)$-norm for $r > 0$ and $\Omega > 1$. Next, we will use the linearity of the space of harmonic functions to form a "Bergman kernel". That is, if $u_1, \ldots, u_{\mathcal{N}}$ are the harmonic functions, set

$$K(x) = \sum_{i=1}^{\mathcal{N}} |u_i(x)|^2 \,. \tag{3.34}$$

Combining some standard linear algebra with the mean value inequality, we obtain pointwise bounds for $K(x)$ depending on Ω and the $L^2(B_{\Omega r} \cap \Sigma)$-norm but independent of $\mathcal{N}$. Integrating this over $B_r \cap \Sigma$ gives a bound on $\mathcal{N}$ which depends on Ω.

To bound the dimension of $\mathcal{H}_d(\Sigma)$ polynomially in d, we choose Ω to be approximately $1 + \frac{1}{d}$. The resulting estimates are then polynomial in d where the degree depends on both the rate of blowup at the boundary of the mean value inequality and the regularity of the volume measure.

See [CM4] for related and more general results in this direction. For results on harmonic functions with polynomial growth in other contexts, see [CM2], [CM5], and the references therein.

3.3 General Constructions

In this section, we will study the growth and independence properties of spaces of functions. To begin with, the following lemma illustrates the usefulness of the assumption on the growth rates of functions.

Lemma 3.14 *Suppose that $f_1, \ldots, f_{2s}$ are nonnegative nondecreasing functions on $(0, \infty)$ such that none of the f_i vanishes identically and for some $d_0\,, K > 0$ and all i*

$$f_i(r) \leq K(r^{d_0} + 1)\,. \tag{3.35}$$

For all $\Omega > 1$, there exist s of these functions $f_{\alpha_1}, \ldots, f_{\alpha_s}$ and infinitely many integers, $m \geq 1$, such that for $i = 1, \ldots, s$

$$f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{2d_0} f_{\alpha_i}(\Omega^m)\,. \tag{3.36}$$

PROOF: Since the functions are nondecreasing and none of them vanish identically, we may suppose that for some $R > 0$ and any $r > R$, $f_i(r) > 0$ for all i.

We will show that there are infinitely many m such that there is some rank s subset of $\{f_i\}$ (where the subset could vary with m) satisfying (3.36). This will suffice to prove the lemma; since there are only finitely many rank s subsets of the $2s$ functions, one of these rank s subsets must be repeated infinitely often.

For $r > R$, note that

$$g(r) = \prod_{i=1}^{2s} f_i(r) \leq K^{2s}(r^{d_0} + 1)^{2s} \,, \tag{3.37}$$

and g is a positive nondecreasing function. Assume that there are only finitely many $m \geq \frac{\log R}{\log \Omega}$ satisfying (3.36). Let $m_0 - 1$ be the largest such m; for all $j \geq 1$ we have that

$$\Omega^{2d_0(s+1)} g(\Omega^{m_0+j-1}) < g(\Omega^{m_0+j}) \,. \tag{3.38}$$

Iterating this and applying (3.37) gives for any $j \geq 1$

$$\Omega^{2d_0(s+1)j} g(\Omega^{m_0}) < g(\Omega^{m_0+j}) \leq \tilde{c} \left(\Omega^j\right)^{2sd_0} \,, \tag{3.39}$$

where $\tilde{c} = \tilde{c}(s, m_0, \Omega, K)$. Since $\Omega > 1$, taking j large yields the contradiction. ∎

For $r > 0$, $B_r = \{|x| < r\} \subset \mathbb{R}^n$, and functions u and v, let

$$I_u(r) = \int_{B_r \cap \Sigma} u^2 \tag{3.40}$$

and

$$J_r(u, v) = \int_{B_r \cap \Sigma} u\, v \,. \tag{3.41}$$

Note that J_r is an inner product and $I(r)$ is the corresponding quadratic form. Furthermore, if $u \in \mathcal{H}_d(\Sigma)$, then $I_u(r) \leq C(1 + r^{2d+k})$.

Given a linearly independent set of functions in $\mathcal{H}_d(\Sigma)$, we will construct functions of one variable which reflect the growth and independence properties of this set.

We begin with two definitions. The first constructs the functions whose growth properties will be studied.

Definition 3.15 ($w_{i,r}$ and f_i) Suppose that $u_1, \ldots, u_s$ are linearly independent functions. For each $r > 0$ we will now define an J_r-orthogonal spanning set $w_{i,r}$ and functions f_i. Set $w_{1,r} = w_1 = u_1$ and $f_1(r) = I_{w_1}(r)$. Define $w_{i,r}$ by requiring it to be orthogonal to $u_j | B_r$ for $j < i$ with respect to the inner product J_r and so that

$$u_i = \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j + w_{i,r} \,. \tag{3.42}$$

Note that $\lambda_{ij}(r)$ is not uniquely defined if the $u_i | B_r$ are linearly dependent. However, since the u_i are linearly independent on Σ, $\lambda_{ij}(r)$ will be uniquely defined for r sufficiently large.

In any case, the following quantity is well-defined for all $r > 0$ (and, in fact, is positive for r sufficiently large)

$$f_i(r) = \int_{B_r} |w_{i,r}|^2 \,. \tag{3.43}$$

In the next proposition, we will record some key properties of the functions f_i from Definition 3.15.

Proposition 3.16 (Properties of f_i) *If $u_1, \dots, u_s \in \mathcal{H}_d(\Sigma)$ are linearly independent, the f_i from Definition 3.15 have the following four properties: There exists a constant $K > 0$ (depending on the set $\{u_i\}$) such that for $i = 1, \dots, s$*

$$f_i(r) \leq K(r^{2d+k} + 1)\,, \tag{3.44}$$

(3.45) *f_i is a nondecreasing function,*

(3.46) *f_i is nonnegative and positive for r sufficiently large, and*

(3.47) *$f_i(r) = I_{w_{i,r}}(r)$ and $f_i(t) \leq I_{w_{i,r}}(t)$ for $t < r$.*

PROOF: Note first that $f_i(r) \leq I_{u_i}(r)$; thus we get (3.44). Furthermore, for $s < r$

$$\begin{aligned} f_i(s) &= \int_{B_s} |u_i - \sum_{j=1}^{i-1} \lambda_{ji}(s)\, u_j|^2 = I_{w_{i,s}}(s) \\ &\leq \int_{B_s} |u_i - \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j|^2 = I_{w_{i,r}}(s) \\ &\leq \int_{B_r} |u_i - \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j|^2 = I_{w_{i,r}}(r) = f_i(r)\,, \end{aligned} \tag{3.48}$$

where the first inequality of (3.48) follows from the orthogonality of $w_{i,r}$ to u_j for $j < i$, and the second inequality of (3.48) follows from the monotonicity of I. From (3.48) and the linear independence of the u_i, we get (3.45) and (3.46). Finally, (3.48) also contains (3.47). ∎

In the following proposition, we will apply Lemma 3.14 to the functions f_i from Definition 3.15:

Proposition 3.17 *Suppose that $u_1, \dots, u_{2s} \in \mathcal{H}_d(M)$ are linearly independent. Given $\Omega > 1$ and $m_0 > 0$, there exist $m \geq m_0$, an integer $\ell \geq \frac{1}{2}\Omega^{-4d-2k}\, s$, and functions $v_1, \dots, v_\ell$ in the linear span of the u_i such that for $i, j = 1, \dots, \ell$*

$$J_{\Omega^{m+1}}(v_i, v_j) = \delta_{i,j}\,, \tag{3.49}$$

and

$$\frac{1}{2}\Omega^{-4d-2k} \leq I_{v_i}(\Omega^m)\,. \tag{3.50}$$

PROOF: By (3.44), (3.45), and (3.47) of Proposition 3.16, applying Lemma 3.14 to the f_i of Definition 3.15 implies that there exist $m \geq m_0$ and a subset $f_{\alpha_1}, \dots, f_{\alpha_s}$ such that for $i = 1, \dots, s$

$$0 < f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{4d+2k} f_{\alpha_i}(\Omega^m)\,. \tag{3.51}$$

Let $w_{\alpha_i,\Omega^{m+1}}$, $i = 1, \dots, s$, be the corresponding functions in the linear span of the u_i as in Definition 3.15.

Consider the s-dimensional linear space spanned by the functions $w_{\alpha_i,\Omega^{m+1}}$ with inner product $J_{\Omega^{m+1}}$. On this space there is also the positive semidefinite bilinear form J_{Ω^m}. Let $v_1, \dots, v_s$ be an orthonormal basis for $J_{\Omega^{m+1}}$ which diagonalizes J_{Ω^m}. We will now evaluate the trace of J_{Ω^m} with respect to these two bases. First, with respect to the orthogonal basis $w_{\alpha_i,\Omega^{m+1}}$ we get by (3.47) and (3.51)

$$s\,\Omega^{-4d-2k} \leq \sum_{i=1}^{s} \frac{I_{w_{\alpha_i},\Omega^{m+1}}(\Omega^m)}{I_{w_{\alpha_i},\Omega^{m+1}}(\Omega^{m+1})}\,. \tag{3.52}$$

Since the trace is independent of the choice of basis we get when evaluating this on the orthonormal basis v_i

$$s\,\Omega^{-4d-2k} \leq \sum_{i=1}^{s} I_{v_i}(\Omega^m)\,. \tag{3.53}$$

Combining this with

$$0 \leq I_{v_i}(\Omega^m) \leq 1\,, \tag{3.54}$$

which follows from the monotonicity of I, we get that there exist at least $\ell \geq \frac{s}{2}\,\Omega^{-4d-2k}$ of the v_i such that for each of these

$$\frac{1}{2}\,\Omega^{-4d-2k} \leq I_{v_i}(\Omega^m) \leq I_{v_i}(\Omega^{m+1}) = 1\,. \tag{3.55}$$

This shows the proposition. ∎

3.4 Finite Dimensionality

In this section, we will show how to bound the dimension of the space of polynomial growth functions which satisfy a mean value inequality. The bound on the dimension will be polynomial in the rate of growth with the exponent determined by the boundary blowup of the mean value inequality.

We will say Σ has the *ϵ-volume regularity property* if for $0 < \epsilon \leq 1$ and $1 \leq C_W < \infty$, given any $0 < \delta \leq \frac{1}{2}$ we get an $R_0 > 0$ such that for all $r \geq R_0$

$$\mathrm{Vol}(B_r \setminus B_{(1-\delta)r} \cap \Sigma) \leq C_W\,\delta^{\epsilon}\ \mathrm{Vol}(B_r \cap \Sigma)\,. \tag{3.56}$$

For example, it follows immediately that any k-dimensional cone has the 1-volume regularity property.

Stationary k-rectifiable varifolds with density bounded above and below have the 1-volume regularity property. To see this, let Σ be a stationary k-rectifiable varifold with density at least 1 almost everywhere and such that

$$\mathrm{V}_\Sigma \equiv \lim_{r\to\infty} \Theta_0(r) < \infty\,. \tag{3.57}$$

Given $0 < \delta \leq \frac{1}{2}$, choose R_0 such that for $R \geq R_0/2$

$$\mathrm{V}_\Sigma - \Theta_0(R) < \delta \, \mathrm{V}_\Sigma \,, \tag{3.58}$$

so that for $R \geq R_0$ we have

$$\begin{aligned} \mathrm{Vol}(B_{(1-\delta)R} \cap \Sigma) &= \Theta_0((1-\delta)R) \; \mathrm{Vol}(B_1 \subset \mathbb{R}^k) \, (1-\delta)^k \, R^k \\ &\geq \mathrm{V}_\Sigma \; \mathrm{Vol}(B_1 \subset \mathbb{R}^k) \, (1-\delta)^{k+1} \, R^k \,. \end{aligned} \tag{3.59}$$

Consequently, for $R \geq R_0$

$$\begin{aligned} \mathrm{Vol}(B_R \setminus B_{(1-\delta)R} \cap \Sigma) &\leq \mathrm{V}_\Sigma \; \mathrm{Vol}(B_1 \subset \mathbb{R}^k) \, \left(1 - (1-\delta)^{k+1}\right) R^k \\ &\leq 2\,(k+1)\,\delta \; \mathrm{Vol}(B_R \cap \Sigma) \,. \end{aligned} \tag{3.60}$$

The following proposition will be used in combination with the results of Section 3.3:

Proposition 3.18 *Let Σ^k be a stationary k-rectifiable varifold with density at least 1 almost everywhere and bounded above by $\mathrm{V}_\Sigma < \infty$. Suppose that $0 < a < 1$ is fixed, $r > 2\,R_0$, and $v_1, \dots, v_{\mathcal{N}}$ are harmonic and J_r-orthonormal. Given any $d \geq 1$ such that for any $R \geq R_0$* (3.60) *holds for any $\delta \geq \frac{1}{4d}$ and for all i*

$$a \leq I_{v_i}((1 - (2d)^{-1})r) \,; \tag{3.61}$$

then

$$\mathcal{N} \leq C \, d^{k-1} \,, \tag{3.62}$$

where $C = \mathrm{V}_\Sigma \, (k+1) \, \frac{k}{k-1} \, a^{-1} \, 2^{k+1}$.

PROOF: Since $d \geq 1$, we can choose a positive integer N with $d \leq N \leq 2d$. For each $x \in B_r$, set

$$K(x) = \sum_{i=1}^{\mathcal{N}} |v_i|^2(x) \,. \tag{3.63}$$

Note that, since each $v_i \in L^2_{loc}(M)$, $K(x)$ must be finite by the mean value inequality. By construction, $K(x)$ is the trace of the symmetric bilinear form

$$(v, w) \to \langle v, w \rangle(x) \tag{3.64}$$

for any v, w in the span of the v_i.

Recall that a symmetric matrix can always be diagonalized by an orthogonal change of basis. Therefore, given $x \in B_r$, we can choose a new orthonormal basis $\{w_i\}$ for the inner product space $(\mathrm{span}\{v_i\}, J_r)$ such that at most one of the w_i, say w_1, is nonvanishing at x. Using the invariance of the trace under orthogonal change of basis, we have that

$$K(x) = \sum_{i=1}^{\mathcal{N}} |w_i|^2(x) = w_1^2(x) \,. \tag{3.65}$$

Now, since each w_i has L^2-norm one on B_r, the mean value inequality gives for $0 < s \leq \frac{1}{2}$ and any $x \in B_{(1-s)r}$

$$|w_i|^2(x) \leq \frac{1}{\mathrm{Vol}(B_{s\,r} \subset \mathbb{R}^k)} \int_{B_{s\,r}(x)} w_i^2 \leq \frac{\mathrm{V}_\Sigma \; s^{-k}}{\mathrm{Vol}(B_r \cap \Sigma)} \,. \tag{3.66}$$

Combining (3.65) and (3.66), we get

$$\mathrm{Vol}(B_r \cap \Sigma)\, K(x) = \mathrm{Vol}(B_r \cap \Sigma)\, |w_1|^2(x) \leq \mathrm{V}_\Sigma \left(\frac{j}{2d}\right)^{-k} , \tag{3.67}$$

for each $j = 1, \ldots, N$ and any $x \in B_{(1-j/(2d))r}$.

We break down the integral of K to get

$$\int_{B_{\left(1-\frac{1}{2d}\right)r}} K = \int_{B_{\left(1-\frac{N}{2d}\right)r}} K + \sum_{j=1}^{N-1} \int_{B_{\left(1-\frac{j}{2d}\right)r} \setminus B_{\left(1-\frac{j+1}{2d}\right)r}} K \,. \tag{3.68}$$

We will now bound each of the two terms in the right-hand side of (3.68). Since $d \leq N \leq 2d$, $B_{(1-N/(2d))r} \subset B_{\frac{r}{2}}$, by (3.67) we have

$$\int_{B_{\left(1-\frac{N}{2d}\right)r}} K \leq \mathrm{V}_\Sigma \left(\frac{1}{2}\right)^{-k} \frac{\mathrm{Vol}(B_{\frac{r}{2}} \cap \Sigma)}{\mathrm{Vol}(B_r \cap \Sigma)} \leq \mathrm{V}_\Sigma \; 2^k \,. \tag{3.69}$$

Bounding the integral of K on each annulus above in terms of its maximum, (3.67) yields

$$\begin{aligned} &\sum_{j=1}^{N-1} \int_{B_{\left(1-\frac{j}{2d}\right)r} \setminus B_{\left(1-\frac{j+1}{2d}\right)r}} K \\ &\quad\leq \mathrm{V}_\Sigma \sum_{j=1}^{N-1} \frac{\mathrm{Vol}(B_{\left(1-\frac{j}{2d}\right)r} \setminus B_{\left(1-\frac{j+1}{2d}\right)r} \cap \Sigma)}{\mathrm{Vol}(B_r \cap \Sigma)} \left(\frac{j}{2d}\right)^{-k} \\ &\quad\leq \mathrm{V}_\Sigma \sum_{j=1}^{N-1} 2(k+1) \left(\frac{1}{2d}\right) \left(\frac{j}{2d}\right)^{-k} , \end{aligned} \tag{3.70}$$

where the second inequality follows from (3.60) (the volume regularity property). Using the elementary inequality

$$\sum_{j=1}^{N-1} j^{-k} = 1 + \sum_{j=2}^{N-1} j^{-k} \leq 1 + \int_1^\infty s^{-k}\, ds = \frac{k}{k-1} \,, \tag{3.71}$$

(3.70) implies that

$$\sum_{j=1}^{N-1} \int_{B_{\left(1-\frac{j}{2d}\right)r} \setminus B_{\left(1-\frac{j+1}{2d}\right)r}} K \leq \mathrm{V}_\Sigma \,(k+1)\, \frac{k}{k-1}\, 2^k\, d^{k-1} \,. \tag{3.72}$$

Substituting (3.69) and (3.72) into (3.68) yields

$$\int_{B_{\left(1-\frac{1}{2d}\right)r}} K \leq \mathrm{V}_\Sigma\, 2^k \left(1 + (k+1)\,\frac{k}{k-1}\,d^{k-1}\right) \leq \mathrm{V}_\Sigma\, 2^{k+1}(k+1)\,\frac{k}{k-1}\,d^{k-1}\,, \tag{3.73}$$

since $d \geq 1$ and $k > 1$.

Combining (3.61) and (3.73), we get

$$\mathcal{N} \leq a^{-1} \int_{B_{\left(1-\frac{1}{2d}\right)r}} K \leq \mathrm{V}_\Sigma\,(k+1)\,\frac{k}{k-1}\,a^{-1}\,2^{k+1}\,d^{k-1}\,. \tag{3.74}$$

∎

Theorem 3.13 now follows directly.

PROOF OF THEOREM 3.13 Set $\Omega = (1 - 1/(2d))^{-1}$ and choose a positive integer m_0 such that $\Omega^{m_0} \geq R_0$. Suppose that $u_1, \ldots, u_{2s} \in \mathcal{H}_d(\Sigma)$ are linearly independent. By Proposition 3.17, there exist $N \geq m_0$, an integer ℓ with

$$\frac{\mathrm{e}^{-4}}{2}\, s \leq \ell\,, \tag{3.75}$$

and J_r-orthonormal functions $f_1, \ldots, f_\ell$ in the linear span of the u_i such that for $i = 1, \ldots, \ell$

$$\frac{\mathrm{e}^{-4}}{2} \leq I_{f_i}((1 - (2d)^{-1})r)\,, \tag{3.76}$$

where we have set $r = \Omega^{N+1}$. Proposition 3.18 (with $a = \mathrm{e}^{-4}/2$) implies that

$$\ell \leq \mathrm{V}_\Sigma\,(k+1)\,\frac{k}{k-1}\;\mathrm{e}^4\;2^{k+2}\,d^{k-1}\,. \tag{3.77}$$

Combining (3.75) and (3.77),

$$\dim \mathcal{H}_d(\Sigma) \leq \mathrm{V}_\Sigma\,(k+1)\,\frac{k}{k-1}\;\mathrm{e}^8\;2^{k+4}\,d^{k-1}\,. \tag{3.78}$$

∎

We will close this chapter with a very brief discussion relating the problems in this chapter to results on minimal cones in Euclidean space.

We leave the proof of the following elementary lemma to the reader (see, for instance, [CM5] for more discussion):

Lemma 3.19 (cf. (2.91) in Chapter 2) *If g is an eigenfunction with eigenvalue λ on N^{k-1} and $p^2 + (k-2)p = \lambda$, then $r^p\, g$ is a harmonic function on $C(N)$. In fact, $r^p\, g \in \mathcal{H}_p(C(N))$.*

As a consequence of this lemma we see that spectral properties of N are equivalent to properties of harmonic functions which grow polynomially on the cone $C(N)$. From this point of view, Theorem 3.13 is roughly an analog of Weyl's asymptotic formula. See [CM1] for further developments on this point of view in a related context.

The spectral properties of spherical minimal submanifolds have been studied in their own right (see, for instance, Cheng-Li-Yau [CgLiYa], Choi-Wang [CiWa] (Theorem 5.7), and Li-Tian [LiTi]).

4

Existence Results

In this chapter, we will discuss the solution to the classical Plateau problem, focusing primarily on its regularity. The first three sections of this chapter cover the basic existence result for minimal disks. After some general discussion of unique continuation and nodal sets, we study the local description of minimal surfaces in a neighborhood of either a branch point or a point of nontransverse intersection. Following Osserman and Gulliver, we rule out interior branch points for solutions of the Plateau problem. In the remainder of the chapter, we prove the embeddedness of the solution to the Plateau problem when the boundary is in the boundary of a mean convex domain. This last result is due to Meeks and Yau.

4.1 The Plateau Problem

The following fundamental existence problem for minimal surfaces is known as the Plateau problem: Given a closed curve Γ, find a minimal surface with boundary Γ. There are various solutions to this problem depending on the exact definition of a surface (parameterized disk, integral current, $\mathbf{Z}_2$ current, or rectifiable varifold). We shall consider the version of the Plateau problem for parameterized disks; this was solved independently by J. Douglas [Do] and T. Rado [Ra1]. The generalization to Riemannian manifolds is due to C. B. Morrey [Mo1].

Theorem 4.1 *Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 closed Jordan curve. Then there exists a piecewise C^1 map u from $D \subset \mathbb{R}^2$ to $\mathbb{R}^3$ with $u(\partial D) \subset \Gamma$ such that the image minimizes area among all disks with boundary Γ.*

The most natural approach to this problem would be to take a sequence of mappings whose areas are going to the infimum and attempt to extract a convergent subsequence. There are two serious difficulties with this.

First, since the area only depends on the image and not the parameterization, the noncompactness of the diffeomorphism group of the disk is a major problem. Namely, let $\phi_k : D \to D$ be a noncompact sequence of diffeomorphisms of the disk and fix $u : D \to \mathbb{R}^3$. The sequence of maps $u(\phi_k)$ has the same image but does not converge.

The second difficulty is that a bound on the area of a map does not give much control on the map. Since long thin tubes can have arbitrarily small area, it is possible to construct a sequence of surfaces in $\mathbb{R}^3$ with boundary $\partial D \subset \mathbb{R}^2 \subset \mathbb{R}^3$ with area close to π whose closure is all of $\mathbb{R}^3$. Namely, we can do the following:

Example 4.2 Let $T(x, y, h, k)$ denote the cylindrical "tentacle" centered at (x, y) of height h and width k (with the top closed off with a disk). This has excess area $\pi\, h\, k$. For each positive integer ℓ, let Σ_ℓ be the topological disk with the tentacles $T(\alpha/2^\ell, \beta/2^\ell, 1, 2^{-4\,\ell})$, $|\alpha|^2 + |\beta|^2 < 2^{2\,\ell}$, attached. The area of Σ_ℓ is less than $\pi + 2^{2\,\ell}(\pi\, 2^{-3\,\ell}) = \pi(1 + 2^{-\ell})$, and hence Σ_ℓ is a minimizing sequence. However, it is obvious that this sequence of surfaces does not converge.

Instead of minimizing area, we will minimize energy (the L^2-norm of the differential of the map). Then we will show that the energy minimizer both minimizes area and finds a good parameterization.

Before giving a proof of this existence result, we shall need a few preliminaries. Let (x, y) be coordinates on $\mathbb{R}^2$ and suppose that $u = (u^1, u^2, u^3)$ is a map from $D \subset \mathbb{R}^2$ to $\mathbb{R}^3$.

If u is in $H^1(D)$, the energy is defined by

$$\mathrm{E}(u) = \int_D |\nabla u|^2 \, dx\, dy = \int_D \left(|u_x|^2 + |u_y|^2\right) \, dx\, dy\,, \tag{4.1}$$

where ∇ is the Euclidean gradient. The area (or two-dimensional Hausdorff measure) of the image of u is

$$\mathrm{Area}(u) = \int_D \left(|u_x|^2\, |u_y|^2 - \langle u_x, u_y\rangle^2\right)^{\frac{1}{2}} \, dx\, dy\,. \tag{4.2}$$

We then have that

$$\mathrm{Area}(u) \leq \frac{1}{2}\, \mathrm{E}(u)\,, \tag{4.3}$$

with equality if and only if $\langle u_x, u_y \rangle$ and $|u_x|^2 - |u_y|^2$ are zero (as L^1 functions). In the case of equality, we say that u is *almost conformal*. If u is an almost conformal immersion, then u is conformal.

One of the key advantages of working in two dimensions is the existence of isothermal coordinates. Namely, given u as above, then there exists a diffeomorphism $\phi : D \to D$ such that $u(\phi) : D \to \mathbb{R}^3$ is almost conformal (see Morrey [Mo2] or Chern [Ch] for a proof). Since the area depends only on the image (and not the parameterization), we have $\mathrm{Area}(u(\phi)) = \mathrm{Area}(u)$.

Remark 4.3 In fact, in theorem 3 of [Mo2], Morrey showed that if ds^2 is a bounded piecewise smooth metric on D, then there exists a homeomorphism $\phi : D \to D$ with $\phi, |\nabla \phi|, |\nabla^2 \phi|$ in L^2 and such that ϕ is almost conformal. This stronger result will be applied later in Theorem 4.35.

Let Γ be a piecewise C^1 closed Jordan curve Γ such that there exists $w : D \to \mathbb{R}^3$ with $w(\partial D) = \Gamma$ and $w \in H^1(D)$. A map $f : \partial D \to \Gamma$ is said to be monotone if the inverse image of every connected set is connected. We define the class

$$X_\Gamma = \{\psi : D \to \mathbb{R}^3 \mid \psi \text{ is piecewise } C^1 \text{ and } \psi|_{\partial D} \text{ is a monotone map onto } \Gamma\}.$$

Define

$$A_\Gamma = \inf_{\psi \in X_\Gamma} \operatorname{Area}(\psi) \text{ and } E_\Gamma = \inf_{\psi \in X_\Gamma} \operatorname{E}(\psi). \tag{4.4}$$

We shall prove Theorem 4.1 in two steps. First, we prove that for each parameterization of the boundary curve Γ there is an energy-minimizing map from the disk with these boundary values. Each such map is harmonic. Since the target is Euclidean, this means that the components u^i are harmonic functions. This is, of course, the standard Dirichlet problem. Second, we will minimize energy over the possible parameterizations of the boundary to obtain *the* energy minimizer which will also solve the Plateau problem. Of course, to extract a limit here we will need uniform estimates for the solutions in the sequence. The key estimate for doing this is the Courant-Lebesgue lemma below.

4.2 The Dirichlet Problem

We will use the following version of the solution to the well-known Dirichlet problem for harmonic maps:

Proposition 4.4 *Let $f : \partial D \to \mathbb{R}^3$ be a piecewise C^1 parameterization of a closed Jordan curve. There exists a unique (energy-minimizing) harmonic map $u : D \to \mathbb{R}^3$ which agrees with f on the boundary and is smooth in the interior.*

Before proving the proposition, we need to recall some preliminaries. $H^1 = H^1(D)$ will denote the Sobolev space of functions f such that f and $|\nabla f|$ are square integrable on the disk D, and the H^1-norm of f is

$$|f|^2_{H^1} = \int_D f^2 + \int_D |\nabla f|^2 . \tag{4.5}$$

We set $C_0^\infty(D)$ equal to the space of smooth functions with compact support on D, and then let $H_0^1(D) \subset H^1(D)$ denote the closure of $C_0^\infty(D)$ with respect to the H^1-norm.

We shall use the following weak compactness result for H^1 functions (see theorem 7.22 of [GiTr] for a proof):

Lemma 4.5 (Rellich Compactness) *Suppose that $u_k : D \to \mathbb{R}^3$ is a sequence of H^1 maps with uniformly bounded H^1-norm; then there exists an H^1 map $u : D \to \mathbb{R}^3$ such that $u_k \to u$ strongly in L^2, $\nabla u_k \to \nabla u$ weakly in L^2, and*

$$\int_D |\nabla u|^2 \leq \liminf \int_D |\nabla u_k|^2 . \tag{4.6}$$

Moreover, if each u_k is in H_0^1, then so is u.

We will also need the following Dirichlet Poincaré inequality for the disk:

Lemma 4.6 (Dirichlet Poincaré Inequality) *There exists a constant $C < \infty$ such that if $u \in H^1_0(D)$,*

$$\int_D u^2 \leq C \int_D |\nabla u|^2 . \tag{4.7}$$

Note that Lemma 4.6 can be proven by a compactness argument using Lemma 4.5. See [GiTr] for a proof of Lemma 4.6.

Recall that a function $v \in H^1(D)$ is said to be weakly harmonic if for every smooth ψ with compact support in D

$$\int_D \langle \nabla v, \nabla \psi \rangle \, dx \, dy = 0 . \tag{4.8}$$

It follows immediately from integration by parts that smooth harmonic functions are weakly harmonic.

We shall also need the following elementary regularity result for harmonic functions on the disk:

Lemma 4.7 (Weyl's Lemma) *Suppose that $v : D \to \mathbb{R}$ is an H^1 function which is weakly harmonic. Then v can be taken to be C^∞ on the interior of D.*

Of course, this result follows from standard regularity theory (see, for instance, theorem 2.10 of [GiTr]), but we shall sketch the proof because of its simplicity. Since the proof relies on convolution with an approximate identity, we briefly recall this technique.

Let $\psi : [0,1] \to \mathbb{R}$ be a smooth nonnegative monotone nonincreasing function with

$$2\,\pi \int_0^1 \psi(t)\, t^{n-1} \, dt = 1 \tag{4.9}$$

such that ψ is identically one on $[0, \frac{1}{3}]$ and has support on $[0, \frac{2}{3}]$. Define $\phi_t : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi_t(x) = t^{-n}\, \psi(|x|/t) . \tag{4.10}$$

Hence, ϕ_t is a nonnegative smooth radially symmetric which is supported on the ball of radius t and has total integral one.

Fix some $t < 1$ for the moment and let $v_t : D_{1-t} \to \mathbb{R}$ be the convolution of v and ϕ_t. That is, given y with $|y| < (1-t)$ we have

$$v_t(y) = \int_{\mathbb{R}^2} v(y + x)\, \phi_t(x) \, dx . \tag{4.11}$$

This integral is well-defined since ϕ_t is supported on a ball of radius t. The importance of this construction is that v_t is smooth. To see this, we do a change of variables $z = y + x$ and write (4.11) as

$$v_t(y) = \int_{\mathbb{R}^2} v(z)\, \phi_t(z - y) \, dz . \tag{4.12}$$

We can now differentiate (4.12) under the integral sign and the smoothness of v_t follows from the smoothness of ϕ_t (see, for instance, section 7.2 of [GiTr] for further discussion).

PROOF: Since v_t is smooth, the lemma will follow once we show that v_t agrees with v.

To see this, we use the radial symmetry of ϕ_t and the mean value property for weakly harmonic functions which follows easily from Proposition 1.11. That is, Proposition 1.11 gives for a weakly harmonic function v and any r

$$\int_{\theta=0}^{2\pi} v(y + r\,\theta)\, d\theta = 2\,\pi\, v(y)\,. \tag{4.13}$$

Writing (4.11) in polar coordinates, we get

$$\begin{aligned} v_t(y) &= \int_{r=0}^{t} \int_{\theta=0}^{2\pi} v(y + r\,\theta)\, \phi_t(r\,\theta)\, d\theta\, dr \\ &= \int_{r=0}^{t} t^{-n}\, \psi\left(\frac{r}{t}\right) \int_{\theta=0}^{2\pi} v(y + r\,\theta)\, d\theta\, dr \\ &= 2\pi\, t^{-n} \int_{r=0}^{t} \psi\left(\frac{r}{t}\right) v(y)\, dr = v(y)\,. \end{aligned} \tag{4.14}$$

∎

Note that the argument actually gives interior estimates for v and all of its derivatives. It is easy to see that the harmonic function is real analytic.

Corollary 4.8 *Suppose that $v : D \to \mathbb{R}$ is a smooth harmonic function; then v is real analytic in the interior of D.*

The above corollary is true much more generally. For ease of exposition, we will give an elementary two-dimensional argument.

PROOF: The complex-valued function $v = u_{x_1} - i\, u_{x_2}$ is smooth and we compute that

$$v_{x_1} = u_{x_1\, x_1} - i\, u_{x_2\, x_1} \tag{4.15}$$

and

$$v_{x_2} = u_{x_1\, x_2} - i\, u_{x_2\, x_2}\,. \tag{4.16}$$

Equations (4.15) and (4.16) imply that v satisfies the Cauchy-Riemann equations and is hence holomorphic. Therefore the real and imaginary parts of v, namely u_{x_1} and u_{x_2}, are real analytic. Integrating this shows that u is itself real analytic. ∎

PROOF OF PROPOSITION 4.4 We will solve the Dirichlet problem for each component f_i of f. In fact, given any piecewise C^1 function w on ∂D we will find a smooth harmonic function v on D which agrees with w on the boundary.

Since w is piecewise C^1 on ∂D, we can extend it to a function $\tilde{w}$ on D which is continuous and in H^1 (see, for instance, section 5.4 of [Ev]). Let H^1_w denote the space of functions in $f \in H^1$ such that $(f - \tilde{w}) \in H^1_0$.

We will see next that the existence of a weak solution follows from the direct method in the calculus of variations. To see this, choose a minimizing sequence of functions $v_\ell \in H^1_w$. The triangle inequality gives

$$\int_D |\nabla(v_\ell - \tilde{w})|^2 \leq 2 \int_D |\nabla v_\ell|^2 + 2 \int_D |\nabla \tilde{w}|^2 \,. \tag{4.17}$$

Since $(v_\ell - \tilde{w}) \in H^1_0$, Lemma 4.6 implies that

$$\int_D (v_\ell - \tilde{w})^2 \leq C \int_D |\nabla(v_\ell - \tilde{w})|^2 \,. \tag{4.18}$$

Combining (4.17), (4.18), the Rellich compactness theorem, and Lemma 4.5, implies that there is an H^1_w function v and a weakly convergent subsequence $v_{\ell'}$ in H^1 such that $\mathrm{E}(v) \leq \liminf \mathrm{E}(v_\ell)$. Therefore $\mathrm{E}(v)$ is equal to the least possible energy (namely, v is energy-minimizing). It remains to check that v is smooth on the interior.

Since v is energy-minimizing, we have for any smooth ψ with compact support in D that

$$\mathrm{E}(v) = \int_D |\nabla v|^2 \, dx \, dy \leq \mathrm{E}(v + t\,\psi) = \int_D \langle \nabla v + t\nabla\psi, \nabla v + t\nabla\psi \rangle \,. \tag{4.19}$$

Differentiating (4.19) at $t = 0$ gives

$$\frac{d}{dt} \mathrm{E}(v + t\,\psi)|_{t=0} = 0 \,, \tag{4.20}$$

and hence for any smooth ψ with compact support in D

$$\int_D \langle \nabla v, \nabla \psi \rangle \, dx \, dy = 0 \,. \tag{4.21}$$

The limit function v is then weakly harmonic and hence, by Lemma 4.7, smooth on the interior of the disk.

Uniqueness follows immediately from the maximum principle. Namely, the difference of two solutions is a harmonic function in $H^1_0(D)$ and thus vanishes identically.

Sobolev space trace theory (see section 5.5 of [Ev]) implies that $H^1(D)$ maps into $L^2(\partial D)$ and hence $v_{\ell'}$ converges to w in $L^2(\partial D)$. Consequently, v is equal to w on the boundary in $L^2(\partial D)$ (in fact, even in $H^{\frac{1}{2}}(\partial D)$). ∎

4.3 The Solution to the Plateau Problem

We will now apply the solution of the Dirichlet problem in the previous section to solve the Plateau problem by minimizing over the possible boundary parameterizations.

Given any point $p \in D$, for each $\rho > 0$ we define $C_\rho = \{q \in D \mid |p-q| = \rho\}$, $d(C_\rho)$ to be the diameter of the image of the curve C_ρ, and $L(C_\rho)$ to be the length of the image of the curve C_ρ.

Lemma 4.9 (Courant-Lebesgue Lemma) *Let u in H^1 be a map from $D \subset \mathbb{R}^2$ to $\mathbb{R}^3$ with $\mathrm{E}(u) \leq K$ that is continuous on $\overline{D}$. For each positive $\delta < 1$, there exists some $\rho \in [\delta, \sqrt{\delta}]$ such that*

$$(d(C_\rho))^2 \leq 2\pi\epsilon_\delta, \tag{4.22}$$

where $\epsilon_\delta = \frac{4\pi K}{-\log\delta} \to 0$ when $\delta \to 0$.

PROOF: By the usual density and regularization arguments (see, for instance, chapter 5 of [Ev]), it suffices to prove the lemma assuming that u is C^1. In this case, we will bound $L(C_\rho)$ and this immediately implies a bound on $d(C_\rho)$.

Define the quantity $p(r)$ by

$$p(r) = r\int_{C_r} |\nabla u|^2\, ds\,, \tag{4.23}$$

where ds is arclength measure on C_r. We have

$$\int_\delta^{\sqrt{\delta}} p(r)\, d(\log r) = \int_\delta^{\sqrt{\delta}} p(r)\,\frac{dr}{r} \leq \int_D |\nabla u|^2\, dx\, dy \leq K\,. \tag{4.24}$$

Consequently, by the mean value theorem in one-variable calculus, there exists some ρ between δ and $\sqrt{\delta}$ such that

$$p(\rho) \leq \left(\int_\delta^{\sqrt{\delta}} p(r)\, d(\log r)\right)\left(\int_\delta^{\sqrt{\delta}} d(\log r)\right)^{-1} \leq \frac{2K}{-\log\delta}\,. \tag{4.25}$$

Given any $r \in [\delta, \sqrt{\delta}]$, the Cauchy-Schwarz inequality gives

$$L(C_r) \leq \int_{C_r} |\nabla u|\, ds \leq (2\pi\, p(r))^{\frac{1}{2}}\,. \tag{4.26}$$

Combining (4.25) and (4.26), we get $(L(C_\rho))^2 \leq \frac{4\pi K}{-\log\delta}$ which completes the proof. ∎

There is one final difficulty to overcome: the noncompactness of the conformal group of D. Recall that a diffeomorphism $\phi : D \to D$ which takes the boundary to the boundary is conformal if it preserves angles. This is equivalent to $\langle\phi_x, \phi_y\rangle = 0$ and $|\phi_x|^2 = |\phi_y|^2$. The noncompactness of the conformal group is a problem since energy is conformally invariant in dimension two.

Lemma 4.10 *Let u be an H^1 function on D and $\phi : D \to D$ a conformal diffeomorphism. Then $\mathrm{E}(u) = \mathrm{E}(u(\phi))$.*

PROOF: Let g_{ij} denote the pullback under ϕ of the Euclidean metric δ_{ij}. The conformality of ϕ gives that

$$g_{ij} = \langle \phi_i, \phi_j \rangle = \frac{1}{2} |\nabla \phi|^2 \delta_{ij} \,. \tag{4.27}$$

Metrics which are related by a scalar factor in this way are said to be conformal. Next, we will observe that energy is invariant under conformal changes of metric. Suppose that $g_{ij} = \lambda^2 \, \delta_{ij}$. We have

$$\begin{aligned} \mathrm{E}_g(u) &= \int_D g^{ij} u_i u_j \, (\det g_{ij})^{\frac{1}{2}} \, dx \, dy \\ &= \int_D \lambda^{-2} \, |\nabla u|^2 \, \lambda^2 \, dx \, dy = \mathrm{E}(u) \,. \end{aligned} \tag{4.28}$$

∎

Recall that the conformal group of D is the group of linear fractional transformations. This group acts triply-transitively on the boundary, that is, given two triples of distinct points on the boundary, there exists a linear fractional transformation taking one to the other. However, the energy and area are both invariant under conformal reparameterizations of the domain; hence, we need to mod out this action. Given three distinct points q_ℓ in Γ and three distinct points p_ℓ in ∂D, we define the class of mappings $\mathcal{X}_-{}' = \{ \psi \in \mathcal{X}_- \mid \psi(p_l) = q_l \}$. It follows that any element of $\mathcal{X}_-$ is conformally equivalent to an element of $\mathcal{X}_-{}'$. The class $\mathcal{X}_-{}'$ has the following well-known equicontinuity property:

Lemma 4.11 *For any constant K, the family of functions*

$$\mathcal{F} = \{ \psi|_{\partial D} \mid \psi \in \mathcal{X}_-{}' \text{ and } \mathrm{E}(\psi) \leq K \} \tag{4.29}$$

is equicontinuous on ∂D. Hence, by the Arzela-Ascoli theorem, $\mathcal{F}$ is compact in the topology of uniform convergence.

PROOF: Let $\psi \in \mathcal{X}_-{}'$ with $\mathrm{E}(\psi) \leq K$. Suppose that we are given $\epsilon > 0$; without loss of generality, we can take ϵ smaller than $\min |q_i - q_j|$. Then, since Γ is a simple closed curve of finite length, it follows that there exists some $d > 0$ such that $|p - q| < d$ for any two distinct points in Γ. This implies that $\Gamma \setminus \{p, q\}$ will have one component of diameter less than ϵ. Choose some $\delta < 1$ such that $\sqrt{2\pi\epsilon_\delta} < d$ and such that given any $p \in \partial D$ at least two of the p_i are not in the ball of radius $\sqrt{\delta}$ about p.

Now, given any $p \in \partial D$, the Courant-Lebesgue lemma, i.e., Lemma 4.9, implies that there is some ρ between δ and $\sqrt{\delta}$ such that $d(C_\rho) < d$. The curve C_ρ divides ∂D into two components, A^1 and A^2, with the larger one (say, A^2) containing at least two of the base points p_i. Denote the corresponding arcs on Γ by $\mathcal{A}^1$ and $\mathcal{A}^2$. Since the image of the endpoints of C_ρ are connected by a curve of length less than d, one of $\mathcal{A}^1$ and $\mathcal{A}^2$ has diameter less than ϵ. Note that this component cannot contain two of the base points q_i, and we can thus conclude that $\mathcal{A}^1$ has diameter less than ϵ. Since $\mathcal{A}^1$ is the image of A^1, this completes the proof of equicontinuity. ∎

The existence of isothermal coordinates gives the following lemma:

Lemma 4.12 *$A_\Gamma = \frac{1}{2}E_\Gamma$ and $D(\psi) = E_\Gamma$ imply that $A(\psi) = A_\Gamma$.*

PROOF: It follows immediately from (4.3) that $A_\Gamma \leq \frac{1}{2}E_\Gamma$.
The other direction follows from the existence of isothermal coordinates (see Remark 4.3). ∎

We are now prepared to solve the Plateau problem.

PROOF OF THEOREM 4.1 By Proposition 4.4 there exists a minimizing sequence $\{u_j\}$ of harmonic maps in $\mathcal{X}_-{}'$ so that $\mathrm{E}(u_j) \to E_\Gamma$. By the compactness result, i.e., Lemma 4.5, there is a weakly convergent subsequence $u_k \to u \in H^1$ with $\mathrm{E}(u) \leq \liminf\{\mathrm{E}(u_k)\}$. Next, the equicontinuity on the boundary, i.e., Lemma 4.11, we can choose the subsequence above such that the functions converge uniformly on the boundary to a continuous function. Therefore, by the lower semicontinuity of energy, $\mathrm{E}(u) = E_\Gamma$ (which also implies that u is harmonic). Finally, Lemma 4.12 implies that $\mathrm{Area}(u) = A_\Gamma$ and that u is pointwise conformal almost everywhere. ∎

Recall that map u is said to be monotone on ∂D if the inverse image of a connected set is connected. For future reference, it is convenient to make the following definition:

Definition 4.13 Let Γ be a piecewise C^1 closed Jordan curve and $u : \bar{D} \to \mathbb{R}^3$ be an almost conformal piecewise C^1 harmonic map whose restriction to ∂D is a monotone map to Γ. If u minimizes energy among all such maps, we will say that u is a *solution of the Plateau problem*.

It remains to discuss the regularity of u up to the boundary ∂D. General arguments for boundary regularity (applied in a wide variety of settings) were given by S. Hildebrandt in [Hi]. Since our primary concern is with the case of minimal surfaces in $\mathbb{R}^3$, we recall the following result of J. C. C. Nitsche [Ni1]:

Theorem 4.14 (Nitsche [Ni1]) *If Γ is a regular Jordan curve of class $C^{k,\alpha}$ where $k \geq 1$ and $0 < \alpha < 1$, then a solution u of the Plateau problem is $C^{k,\alpha}$ on all of $\bar{D}$.*

The book [Ni2] contains a proof of Theorem 4.14 and an extensive discussion of boundary regularity.

4.4 Preliminary Discussion of Branch Points

We will next show that the solution to the Plateau problem given in the previous section is, in fact, immersed. To see this, we first show that branch points are isolated (which is true for any minimal surface). Then, using the minimizing property, we show that interior branch points do not exist. It follows that the solution is an immersed surface.

If we let $z = (x_1 + i\, x_2)$ be the complex coordinate on D, the harmonicity of u implies that $v \equiv u_{x_1} - i\, u_{x_2}$ is a (vector-valued) holomorphic function. To see this, we differentiate

$$v_{\bar z} = (u_{x_1} - i\, u_{x_2})_{x_1} + i(u_{x_1} - i\, u_{x_2})_{x_2} = (u_{x_1\, x_1} + u_{x_2\, x_2}) + i(u_{x_1\, x_2} - u_{x_2\, x_1}) = 0\,. \tag{4.30}$$

Furthermore, we have

$$|v|^2 = |u_{x_1}|^2 + |u_{x_2}|^2\,, \tag{4.31}$$

and hence v vanishes precisely at the branch points of u. Since the zeros of a nonconstant holomorphic function on D must be isolated, this gives the following:

Corollary 4.15 *An almost conformal harmonic map $u : D \to \mathbb{R}^3$ has isolated branch points.*

In fact, one can prove quite a bit more. R. Osserman showed that the image surface was immersed (namely, that there are no interior geometric branch points). R. Gulliver proved the definitive result: The solution has no interior branch points (so that not only is the image an immersed surface, but the parameterization does not introduce any branch points); cf. H. W. Alt. We will prove these results in the next three sections.

4.5 Unique Continuation

In studying the local properties of minimal surfaces, it will be useful to have a good description of their possible self-intersections and branch points. This can be achieved by using results about strong unique continuation for solutions of elliptic equations. The results of the next two sections will be used in the following sections to show first that minimal disks are immersed and then that (under additional hypotheses) they are embedded.

In particular, we will need the following quantitative version of strong unique continuation:

Theorem 4.16 (Quantitative Strong Unique Continuation) *Suppose that v is a nonconstant solution on D to*

$$0 = (a_{i,j}\, v_{x_j})_{x_i} + b_i\, v_{x_i} + c\, v\,, \tag{4.32}$$

where a_{ij} is symmetric, uniformly elliptic, and Lipschitz, and b_i, c are continuous. Then there exists some $\epsilon = \epsilon(a_{ij}, b_i, c) > 0$ and $\bar d$ such that for any $2t < \epsilon$

$$\frac{\int_{|x|=2t} v^2}{\int_{|x|=t} v^2} \leq 2^{2\,\bar d + 1}\,. \tag{4.33}$$

In particular, there exists some integer $d \leq \bar d$ such that v vanishes precisely to order d at the origin.

In fact, it will follow from the proof of Theorem 4.16 that $\bar{d}$ may be estimated from above in terms of the growth of v from D_ϵ to $D_{1/2}$.

Remark 4.17 In particular, Theorem 4.16 implies that v cannot vanish to infinite order at any point. An operator which has this property is said to satisfy *strong unique continuation*. Note that if L has this strong unique continuation property, $Lv = 0$ on a connected domain Ω, and v vanishes on an open subset of Ω, then necessarily $v \equiv 0$ on Ω.

In the case where $a_{ij} = \delta_{ij}$ and $b_i = c = 0$, this theorem is closely related to Hadamard's three circles theorem from complex analysis. Theorem 4.16 is, by now, standard; for instance, a proof can be found in [GaLi]. We will give a proof in the case of the Laplacian on $\mathbb{R}^2$. This proof can easily be modified to work in the general case.

Given a nonconstant function v with $\Delta v = 0$ on D, we will next define $I(t)$, $E(t)$, and the frequency function $U(t)$ for $0 < t \leq 1$ (cf. Almgren [Am2]). First, define $I(t)$ by

$$I(t) = t^{-1} \int_{|x|=t} v^2 = \int_{\theta=0}^{2\pi} v^2(t\cos\theta, t\sin\theta)\, d\theta \,. \tag{4.34}$$

Next, define $E(t)$ by

$$E(t) = \int_{|x|\leq t} |\nabla v|^2 = \int_{|x|=t} v_r\, v \,, \tag{4.35}$$

where $v_r = \frac{\partial v}{\partial r}$. Note that the second equality follows from Stokes' theorem since $\Delta v^2 = 2|\nabla v|^2$. Finally, define the *frequency function* $U(t)$ by

$$U(t) = \frac{E(t)}{I(t)} \,. \tag{4.36}$$

Lemma 4.18 *If v is harmonic on D and $0 < t \leq 1$, then*

$$E'(t) = \int_{|x|=t} |\nabla v|^2 = 2 \int_{|x|=t} v_r^2 \,. \tag{4.37}$$

In this case (where v is harmonic on the two-dimensional disk), Lemma 4.18 can be proven by writing $v = \operatorname{Re} f$ (where f is holomorphic), using the Cauchy-Riemann equations, and integrating by parts. However, we will give a proof which can be easily modified to apply more generally (we will return to this point).

PROOF: The coarea formula (i.e., (1.40)) gives

$$E'(t) = \int_{|x|=t} |\nabla v|^2 = t^{-1} \int_{|x|=t} \langle |\nabla v|^2 x, |x|^{-1}\, x \rangle \,. \tag{4.38}$$

Since

$$\operatorname{div}(|\nabla v|^2 x) = (v_{x_i}^2 x_j)_{x_j} = 2 x_j v_{x_i} v_{x_i x_j} + 2 v_{x_i}^2 \,, \tag{4.39}$$

applying Stokes' theorem to (4.38) gives

$$E'(t) = 2t^{-1}E(t) + 2t^{-1} \int_{|x|\le t} x_j\, v_{x_i} v_{x_i\, x_j}\,, \tag{4.40}$$

where by convention we sum over repeated indices. Since $\Delta v = v_{x_i x_i} = 0$,

$$x_j v_{x_i} v_{x_i x_j} = (x_j v_{x_i} v_{x_j})_{x_i} - v_{x_i}^2\,. \tag{4.41}$$

Substituting (4.41) into (4.40) and integrating by parts, we get

$$E'(t) = 2\,t^{-2} \int_{|x|=t} (x_j v_{x_j})^2 = 2 \int_{|x|=t} v_r^2\,. \tag{4.42}$$

∎

Example 4.19 Suppose that $v(x)$ is a homogeneous harmonic polynomial of degree d. It is easy to see that $r\,v_r = d\,v$ and hence that $E(t) = d\,I(t)$. Consequently, in this case, $U(t) \equiv d$. More generally, suppose that $w = w_1 + w_2$ is a harmonic function where w_1 is homogeneous of degree d and the Taylor series of w_2 starts at degree $d+1$. Using the orthogonality of the spherical harmonics, we see that $U(t) \ge d$ in this case.

Lemma 4.20 *If v is harmonic on D, then the frequency function $U(t)$ is monotone nondecreasing.*

PROOF: From the definition (4.36),

$$(\log U)'(t) = \frac{E'(t)}{E(t)} - \frac{I'(t)}{I(t)}\,. \tag{4.43}$$

Consequently, it suffices to show that

$$I'(t)\,E(t) \le E'(t)\,I(t)\,. \tag{4.44}$$

By Stokes' theorem, differentiating (4.34) gives

$$I'(t) = 2t^{-1} \int_{|x|=t} v\,v_r = 2\frac{E(t)}{t}\,. \tag{4.45}$$

Multiplying both sides of (4.45) by $E(t)$ and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} I'(t)\,E(t) &= \frac{2}{t}E(t)^2 = \frac{2}{t}\left(\int_{|x|=t} v\,v_r\right)^2 \\ &\le 2\,t^{-1} \int_{|x|=t} v^2 \int_{|x|=t} v_r^2 = I(t)\,E'(t)\,, \end{aligned} \tag{4.46}$$

where the last equality follows from Lemma 4.18. This proves (4.44) and the lemma follows. ∎

Although we will not use it, we note that if v is harmonic and the frequency is constant, then v must be homogeneous and hence a spherical harmonic. Namely, we have the following:

Lemma 4.21 *If v is harmonic on D and the frequency function $U(t)$ is constant (say $U(t) = d$), then v is a homogeneous harmonic polynomial of degree d.*

PROOF: Since $U(t) = d$ is constant, we have equality in the Cauchy-Schwarz inequality in (4.46) and therefore

$$v_r(x) = f(|x|)\, v(x) \tag{4.47}$$

for some function f. Substituting (4.47) into the definition of $U(t)$,

$$d\,t^{-1} \int_{|x|=t} v^2 = d\,I(t) = E(t) = \int_{|x|=t} v\, v_r = f(t) \int_{|x|=t} v^2 \tag{4.48}$$

and hence $f(t) = d\,t^{-1}$. Therefore, (4.47) gives

$$v_r(x) = d\,|x|^{-1}\, v(x)\,. \tag{4.49}$$

It follows from (4.49) that v is homogeneous of degree d and is hence a spherical harmonic. ∎

We are now prepared to prove Theorem 4.16 when $\Delta v = 0$. In fact, we will prove a stronger statement in this case which will cover Theorem 4.27 stated below.

PROOF OF THEOREM 4.16 FOR $\Delta v = 0$ We will show this from the monotonicity of the frequency function. Differentiating $\log I(t)$ (see (4.45)) gives

$$(\log I)'(t) = 2\frac{U(t)}{t}\,. \tag{4.50}$$

Therefore, using the monotonicity of $U(t)$, we can write

$$\frac{I(1)}{I(\frac{1}{2})} = \mathrm{e}^{2\int_{\frac{1}{2}}^{1} \frac{U(t)}{t}\, dt} \geq \mathrm{e}^{2U(\frac{1}{2})\log 2} = 4^{U(\frac{1}{2})}\,. \tag{4.51}$$

Again using the monotonicity of U, we see by (4.51) that $U(t)$ is uniformly bounded for $t \leq \frac{1}{2}$. Consequently, integrating (4.50) and using the uniform bound on $U(t)$, we see that for $0 < 2s \leq t < \frac{1}{4}$

$$\frac{I(2s)}{I(s)} = \mathrm{e}^{2\int_{s}^{2s} \frac{U(t')}{t'}\, dt'} \leq 2^{2U(2s)} \leq \frac{I(2t)}{I(t)}\,. \tag{4.52}$$

The estimate (4.52) is known as a doubling estimate. Iterating (4.52) bounds the order of vanishing at the origin (which gives quantitative strong unique continuation). Consequently, the Taylor polynomial for v at the origin does not vanish identically; let v^1 denote the first nonzero Taylor polynomial (which has degree $d \geq 1$). Since v is harmonic, v_1 and $v - v_1$ are harmonic. By construction, $v - v_1$ vanishes to order at least $d + 1$ and the theorem follows. ∎

The key to proving this strong unique continuation theorem was obtaining the doubling estimate (4.52). This doubling estimate was an immediate consequence of a uniform bound on the frequency. When $\Delta v = 0$ on the disk, we deduced this bound on the frequency from the monotonicity of the frequency. For more general equations, as in Theorem 4.16, we can write down a frequency function $U(t)$, but it is not necessarily monotone. However, if the coefficients are sufficiently regular, then there will exist some constant Λ such that $e^{\Lambda\, t}\, U(t)$ is monotone. This gives enough control to establish a uniform bound on $U(t)$ for small t and hence the doubling estimate.

As an example, consider the case where $a_{ij} = f\, \delta_{ij}$ with $f(0) = 1$, $|\nabla f| \leq \lambda$, and $b_i = c = 0$. Suppose that v is a nonconstant solution on D to

$$\text{div}(f\, \nabla v) = 0\,. \tag{4.53}$$

It will be clear to the reader how to modify these arguments when b_i and c do not vanish. Furthermore, equations of the form (4.53) (including the cases where b_i and c appear) will suffice for all of our applications.

Remark 4.22 As noted at the end of section 2 of [GaLi], equations of the form (4.32) can be transformed into the form (4.53) by making a suitable change of metric (see [GaLi] for details).

In general, the above transformation involves introducing a new metric and hence perturbation terms. In two dimensions, this can be avoided by using the existence of isothermal coordinates.

Example 4.23 (Laplacian for a Riemannian Metric) If g_{ij} is a Riemannian metric in a coordinate chart in $\mathbb{R}^2$, the associated Laplacian Δ_g is given by

$$\Delta_g v = \frac{1}{\sqrt{g}}\, (\sqrt{g}\, g^{ij}\, v_{x_j})_{x_i}\,, \tag{4.54}$$

where g and g^{ij} are the determinant and inverse matrix of g_{ij}. The existence of isothermal coordinates allows us, after a change of coordinates, to suppose that $g_{ij} = f\, \delta_{ij}$ for a positive function f. In this case, (4.54) becomes

$$\Delta_g v = \frac{1}{f}\, v_{x_i\, x_i}\,. \tag{4.55}$$

For $0 < t \leq 1$, define $I(t)$ and $E(t)$ by

$$I(t) = t^{-1} \int_{|x|=t} f\, v^2\,. \tag{4.56}$$

Using Stokes' theorem and the formula $\text{div}(f\, \nabla v^2) = 2f\, |\nabla v|^2$ (by (4.53)), we get

$$E(t) = \int_{|x|\leq t} f\, |\nabla v|^2 = \int_{|x|=t} f\, v_r\, v\,. \tag{4.57}$$

Once again, we define the frequency U as in (4.36).

Henceforth, we assume that $t, \, |x| \leq \frac{1}{2\lambda}$ so that $\frac{1}{2} \leq f(x) \leq \frac{3}{2}$. Differentiating $I(t)$ gives

$$I'(t) = 2t^{-1} \int_{|x|=t} f\, v\, v_r + t^{-1} \int_{|x|=t} f_r\, v^2 \, , \tag{4.58}$$

so Stokes' theorem and the bound $|\nabla f| \leq 2\lambda\, f$ give

$$\left| I'(t) - 2\frac{E(t)}{t} \right| \leq 2\,\lambda\, I(t)\, . \tag{4.59}$$

The definition of $U(t)$ and (4.59) give the differential inequality

$$\left| (\log I)'(t) - 2\frac{U(t)}{t} \right| \leq 2\,\lambda\, . \tag{4.60}$$

The coarea formula (i.e., (1.40)) gives

$$E'(t) = \int_{|x|=t} f\, |\nabla v|^2\, . \tag{4.61}$$

Applying Stokes' theorem to (4.61) as before, we get

$$\begin{aligned} E'(t) &= \int_{|x|=t} f\, |\nabla v|^2 = t^{-1} \int_{|x|=t} \langle f\, |\nabla v|^2 x, |x|^{-1}\, x \rangle \\ &= 2t^{-1} E(t) + t^{-1} \int_{|x|\leq t} x_j\, (f v_{x_i} v_{x_i})_{x_j}\, , \end{aligned} \tag{4.62}$$

since

$$\operatorname{div}(f\, |\nabla v|^2 x) = (f\, v_{x_i}^2 x_j)_{x_j} = x_j\, (f v_{x_i} v_{x_i})_{x_j} + 2f\, v_{x_i}^2\, . \tag{4.63}$$

Since $(f v_{x_i})_{x_i} = 0$, $v_{x_i\, x_j} = v_{x_j\, x_i}$, and $(x_i)_{x_j} = \delta_{ij}$,

$$\begin{aligned} x_j (f v_{x_i} v_{x_i})_{x_j} &= x_j f_{x_j} v_{x_i} v_{x_i} + 2 x_j f v_{x_i} v_{x_j\, x_i} \\ &= x_j f_{x_j} v_{x_i} v_{x_i} + 2 (x_j f v_{x_i} v_{x_j})_{x_i} - 2 x_j v_{x_j} (f v_{x_i})_{x_i} - 2 f v_{x_i}^2 \\ &= x_j f_{x_j} v_{x_i} v_{x_i} + 2 (x_j f v_{x_i} v_{x_j})_{x_i} - 2 f v_{x_i}^2\, . \end{aligned} \tag{4.64}$$

Substituting equation (4.64) into (4.62), integrating by parts, and using the bound $|\nabla f| \leq 2\,\lambda\, f$ yields

$$\left| E'(t) - 2 \int_{|x|=t} f\, v_r^2 \right| \leq 4\,\lambda\, E(t)\, . \tag{4.65}$$

Combining (4.60) and (4.65), differentiating $\log U(t)$ as in (4.43) gives

$$\left| (\log U)'(t) - \frac{2}{E(t)} \int_{|x|=t} f\, v_r^2 + 2\frac{U(t)}{t} \right| \leq 6\,\lambda\, . \tag{4.66}$$

Arguing exactly as in (4.46) and using Cauchy-Schwarz gives

$$0 \leq \frac{2}{E(t)} \int_{|x|=t} f\, v_r^2 - 2\frac{U(t)}{t}\,. \tag{4.67}$$

Combining (4.66) and (4.67), we see that $e^{6\,\lambda\, t}\, U(t)$ is monotone nondecreasing. In particular, for all $0 < t \leq \frac{1}{2\,\lambda}$

$$U(t) \leq e^{6\,\lambda\, t}\, U(t) \leq e^3\, U\left(\frac{1}{2\,\lambda}\right)\,. \tag{4.68}$$

Combining the uniform bound on the frequency, (4.68), and the differential inequality (4.60), we obtain the desired bound on the order of vanishing of v. This completes the proof of Theorem 4.16 for equations of the form (4.53).

4.6 Local Description of Nodal and Critical Sets

Using the results of the previous section, we will obtain a useful local description of solutions to elliptic equations on the disk D. We will focus primarily on obtaining local asymptotic expansions and on the consequences of these expansions for the nodal and critical sets.

Definition 4.24 (Nodal and Critical Sets) Given a function u, the *nodal set* of u is the set $\{u = 0\}$ where u vanishes. The *critical set* of u is the set $\{u = 0\} \cap \{\nabla u = 0\}$ where u and ∇u both vanish.

Before we get to the main results of this section, we will need some background material on elliptic estimates. As we saw in Chapter 1, the mean value inequality bounds the maximum of a harmonic function by its average on a larger ball. Combining this with the reverse Poincaré inequality, one can then control all higher derivatives similarly. For more general equations, similar arguments give the following lemma (see theorem 8.8 of [GiTr] or [HaLi] for a proof):

Lemma 4.25 *Let v be a solution to* (4.32) *on D, and suppose further that a_{ij}, b_i, c are smooth. Then there exists constants $\epsilon > 0$ and $C < \infty$ which depend on a_{ij}, b_i, c such that for $|x| \leq \epsilon$*

$$|v(x)|^2 + |x|^2\, |\nabla v(x)|^2 + |x|^4\, |\nabla^2 v(x)|^2 \leq C\, |x|^{-2} \int_{B_{2\,|x|}} v^2\,. \tag{4.69}$$

Combining Theorem 4.16 and Lemma 4.25 gives the following corollary:

Corollary 4.26 *Let v be a solution to* (4.32) *on D, and let $\bar{d}$ be the bound on the order of vanishing given by Theorem 4.16. Suppose further that a_{ij}, b_i, and c are smooth. Then there exists constants $\epsilon > 0$ and $C < \infty$ which depend on a_{ij}, b_i, c and $\bar{d}$ such that for $|x| \leq \epsilon$*

$$|v(x)|^2 + |x|^2\, |\nabla v(x)|^2 + |x|^4\, |\nabla^2 v(x)|^2 \leq C\, I(|x|)\,. \tag{4.70}$$

PROOF: By Theorem 4.16, the frequency $U(t)$ is uniformly bounded by $\bar{d}$ on some ball $\{|x| \leq 2\,\epsilon\}$. Consequently, we have for any $s \leq \epsilon$ that

$$I(2s) \leq 2^{2\bar{d}}\, I(s)\,. \tag{4.71}$$

Integrating the bound (4.71) (and using the trivial monotonicity of $I(t)$) gives

$$\int_{|x|\leq 2s} v^2 = \int_{t=0}^{2s} \int_{|x|=t} v^2\, dt \leq \int_{t=0}^{2s} t\, I(t)\, dt \leq C_1\, s^2\, I(s)\,. \tag{4.72}$$

The estimate (4.70) now follows by combining (4.69) and (4.72). ∎

Theorem 4.27 (Local Asymptotics [Ber]) *Let v be a solution to* (4.32)*, let $\bar{d}$ and d be the bound on the order of vanishing and the order of vanishing at 0, respectively, given by Theorem 4.16. Suppose further that a_{ij}, b_i, and c are smooth. There exists a linear transformation T such that after rotating by T we can write*

$$v(x) = p_d(x) + q(x)\,, \tag{4.73}$$

where p_d is a homogeneous harmonic polynomial of degree d and

$$|q(x)| + |x|\,|\nabla q(x)| + \cdots + |x|^d\,|\nabla^d q(x)| \leq C|x|^{d+1} \tag{4.74}$$

for some $C < \infty$ which depends on a_{ij}, b_i, c and $\bar{d}$.

PROOF: After a linear transformation, we may assume that $a_{ij}(0) = \delta_{ij}$ and $a_{ij}(x) - \delta_{ij} = c_{ij}(x)$ where $|c_{ij}(x)| \leq \lambda\,|x|$. We can then write (4.32) as

$$\begin{aligned} \Delta v &= -(c_{i,j}\, v_{x_j})_{x_i} - b_i\, v_{x_i} - c\, v \\ &= -c_{i,j}\, v_{x_i\, x_j} - b_i'\, v_{x_i} - c\, v\,, \end{aligned} \tag{4.75}$$

where $b_i' = b_i + (c_{i,j})_{x_j}$.

By Theorem 4.16, $v(x)$ vanishes precisely to order d at the origin. Therefore the limit

$$\lim_{t \to 0} t^{-2d}\, I(t) \tag{4.76}$$

exists and is not zero. In fact, if $p_d(x)$ denotes the degree d Taylor polynomial for v at the origin, then

$$\lim_{t \to 0} t^{-2d}\, I(t) = \lim_{t \to 0} t^{-2d-1} \int_{|x|=t} p_d^2\,. \tag{4.77}$$

Define the error term q by $q(x) = v(x) - p_d(x)$ so that q vanishes to order at least $d+1$ at the origin.

Since v satisfies (4.75), we have

$$\lim_{|x| \to 0} |x|^{d-2} \Delta v = -\lim_{|x| \to 0} |x|^{d-2} \left(c_{i,j}\, v_{x_i\, x_j} - b_i'\, v_{x_i} - c\, v \right)\,. \tag{4.78}$$

The bound on the frequency implies that the estimate (4.70) applies to v. Hence the limits of the second two terms in (4.78) are both zero and $v_{x_i\,x_j}$ is of order $|x|^{d-2}$. However, $|c_{ij}| \leq \lambda\,|x|$ so the last term on the right-hand side also goes to zero. Consequently,

$$|\Delta v| \leq C\,|x|^{d-1}\,. \tag{4.79}$$

On the other hand, since $v - p_d$ is of order at least $d+1$ the Hessian $\nabla^2(v-p_d)$ is of order at least $d-1$. Therefore (4.79) implies that Δp_d is of order at least $d-1$.

However, p_d is homogeneous of degree d and hence Δp_d is a homogeneous degree $d-2$ polynomial. A homogeneous degree $d-2$ polynomial which is of order at least $d-1$ must vanish identically, and hence p_d is a spherical harmonic.

Substituting the definition of q into (4.75), we see that q itself satisfies an elliptic equation

$$\Delta q = \Delta(v - p_d) = \Delta v = -c_{i,j}\,v_{x_i\,x_j} - b_i'\,v_{x_i} - c\,v\,. \tag{4.80}$$

As we already verified, the right-hand side of (4.80) vanishes to order at least $d+1$. Lemma 4.25 then yields the higher-order estimates in (4.74). ∎

We will use Theorem 4.27 repeatedly to describe the local behavior of minimal surfaces and their almost conformal parameterizations.

In two dimensions, a spherical harmonic $p_d(x)$ of degree d can be written as a constant times the real part of $\mathrm{e}^{i\,\theta}\,(x_1 + i\,x_2)^d$ for some $\theta \in [0, 2\pi)$. The real part of a complex vector-valued function f will be denoted by $\mathrm{Re}\,f$.

Lemma 4.28 *Suppose that*

$$v(x) = p_d(x) + q(x)\,, \tag{4.81}$$

where p_d is a nontrivial homogeneous harmonic polynomial of degree d, and there is a constant C such that

$$|q(x)| + |x|\,|\nabla q(x)| \leq C|x|^{d+1}\,; \tag{4.82}$$

then there exists a neighborhood U of 0 and a C^1 diffeomorphism $F : U \to D$ with $F(0) = 0$ such that $F(U \cap \{v = 0\})$ is equal to the $2d$ smooth arcs given by

$$\alpha_j(t) = t\,\mathrm{e}^{\frac{j\,\pi\,i}{d}} \tag{4.83}$$

for $t \in [0, \epsilon]$ and $j = 1, \ldots, 2d$.

PROOF: Since we are in two dimensions, $p_d(x)$ can be written as a constant times the real part of $\mathrm{e}^{i\,\theta}\,z^d$ where $z = x_1 + i\,x_2$. After a rotation, we may suppose that $\theta = 0$ so that

$$v(z) = \mathrm{Re}(z^d + q(z))\,. \tag{4.84}$$

The estimate (4.82) implies that the map

$$F(z) = z(1 + z^{-d}\, q(z))^{\frac{1}{d}} \tag{4.85}$$

is well-defined for $|z| \leq \frac{1}{2C}$. The complex derivative of F is given by

$$\begin{aligned} F_z(z) &= (1 + z^{-d}\, q(z))^{\frac{1}{d}} \\ &\quad + \frac{z}{d}(1 + z^{-d}\, q(z))^{\frac{1-d}{d}} (z^{-d}\, q_z(z) - d\, z^{-d-1}\, q(z))\,. \end{aligned} \tag{4.86}$$

Applying (4.82) again, we see that F is C^1 and $F_z(0) = 1$. The implicit function theorem gives the desired neighborhoods and a C^1 inverse F^{-1}.

Composing with the diffeomorphism F, we see that $v(x) = \mathrm{Re}(F(z))^d$. The description (4.83) follows immediately. ∎

The first application of Theorem 4.27 of Bers will be to obtain a result of Rado.

Theorem 4.29 (Rado [Ra2]) *Suppose that $\Omega \subset \mathbb{R}^2$ is a convex subset and $\sigma \subset \mathbb{R}^3$ is a simple closed curve which is graphical over $\partial\Omega$. Then any minimal disk $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \sigma$ must be graphical over Ω and hence unique by the maximum principle.*

PROOF: Suppose that Σ is such a minimal disk that is not graphical. Then the projection to the (x_1, x_2)-plane cannot be an immersion. Consequently, there exists some $x \in \Sigma$ such that at x $\nabla_\Sigma x_1$ and $\nabla_\Sigma x_2$ are linearly dependent. In other words, there exists $(a, b) \neq (0, 0)$ such that

$$\nabla_\Sigma (a\, x_1 + b\, x_2)(x) = 0\,. \tag{4.87}$$

By Proposition 1.6, $a\, x_1 + b\, x_2$ is harmonic on Σ (since it is a linear combination of coordinate functions). We may therefore apply Theorem 4.27, and it follows from (4.87) and Lemma 4.28 that the nodal line

$$a\, x_1 + b\, x_2 = (a\, x_1 + b\, x_2)(x) \tag{4.88}$$

has a singularity at x where at least four different curves meet. If two of these nodal curves were to meet again, then there would be a closed nodal curve which must bound a disk (since Σ is a disk). By the maximum principle, $a\, x_1 + b\, x_2$ would have to be constant on this disk and hence constant on Σ by unique continuation. This would imply that $\sigma = \partial\Sigma$ is contained in the plane given by (4.88). Since this is impossible, we conclude that all of these curves go to the boundary without intersecting again.

In other words, the plane in $\mathbb{R}^3$ given by (4.88) intersects σ in at least four points. However, since $\Omega \subset \mathbb{R}^2$ is convex, $\partial\Omega$ intersects the line given by (4.88) in exactly two points. Finally, since σ is graphical over $\partial\Omega$, σ intersects the plane in $\mathbb{R}^3$ given by (4.88) in exactly two points, which gives the desired contradiction. ∎

We will next apply Theorem 4.27 to obtain a normal form for a harmonic map $u : D \to \mathbb{R}^3$.

Lemma 4.30 *Let $u : D \to \mathbb{R}^3$ be a nonconstant harmonic map with $u(0) = 0$ and $\nabla u(0) = 0$. The following local asymptotic expansion holds:*

$$u(x) = \mathrm{Re} \sum_{j=d}^{2d-1} a_j (x_1 + i x_2)^j + q(x)\,, \tag{4.89}$$

where $d \geq 2$, each $a_i \in \mathbf{C}^3$, $a_d \neq 0$, and

$$|q(x)| + |x|\,|\nabla q(x)| \leq C|x|^{2d}\,. \tag{4.90}$$

PROOF: Since the components of u are actually harmonic functions on D, Theorem 4.16 implies that u vanishes to some finite order $d \geq 2$ there. Let u^1 denote the first d terms in this expansion (so that u^1 has terms of degree d to $2d-1$ and the degree d term is nontrivial). We can then write $u(x) = u^1(x) + q(x)$ where $q(x)$ satisfies (4.90), and the claim follows. ∎

Suppose now that $u : D \to \mathbb{R}^3$ is an almost conformal harmonic map (so that the image is minimal). We saw in the previous section that u is an immersion away from the (isolated) branch points where u_{x_1} and u_{x_2} both vanish. We are interested in giving a description of u near such a branch point. Our presentation will roughly follow the arguments of Gulliver in [Gu].

We let $z = x_1 + i x_2$ be the complex coordinate on D. Since u is almost conformal,

$$\langle u_z, u_z \rangle = 0\,. \tag{4.91}$$

Note that this gives two equations (namely, that the real and imaginary parts both vanish). Applying this to the local representation (4.89), we get that $\langle a_d, a_d \rangle = 0$ and $\langle a_d, a_{d+1} \rangle = 0$. This first condition implies that the real and imaginary parts of a_d are orthogonal and of equal length. After a rotation, we can assume that $a_d = (a, -i\,a, 0)$ where $a > 0$. After dilating, we can take $a = 1$ so that $a_d = (1, -i, 0)$. Since $\langle a_d, a_{d+1} \rangle = 0$, we can write $a_{d+1} = (c, -i\,c, c')$. Substituting this back into (4.90), we get

$$u_1(z) + i u_2(z) = z^d + c z^{d+1} + G(z)\,, \tag{4.92}$$

where $d \geq 2$ and

$$|z|\,|u_3(z)| + |G(z)| + |z|\,|\nabla G(z)| \leq C\,|z|^{d+2}\,. \tag{4.93}$$

Equations (4.92) and (4.93) give a sort of normal form for minimal surfaces near a branch point. It will be convenient to obtain an even more canonical representation by choosing a nice reparameterization of D.

Lemma 4.31 *Suppose that* (4.92) *and* (4.93) *hold. Then there exist neighborhoods U and V of $0 \in D$ and a C^1 diffeomorphism $F : V \to U$ such that for $z \in V$*

$$u_1 + iu_2 = (F(z))^d \tag{4.94}$$

and $u_3 = \phi(F(z))$, where

$$|\phi(z)| + |z|\,|\nabla\phi(z)| + |z|^2\,|\nabla^2\phi(z)| \le C|z|^{d+1}\,. \tag{4.95}$$

If u is real analytic, then F and ϕ are real analytic away from 0.

Note that C in (4.95) is not the same as in (4.93). The fact that we can take F and ϕ to be real analytic will not be used until we study false branch points.

PROOF: The estimate (4.93) on $G(z)$ implies that for small z

$$|z|^{-d}\,|G(z)| < 1\,. \tag{4.96}$$

We can therefore define

$$F(z) = z(1 + z^{-d}\,G(z))^{\frac{1}{d}}\,. \tag{4.97}$$

Clearly, with this definition (4.94) is satisfied.

The estimate (4.93) implies that F is C^1 and the derivative at the origin is the identity. Consequently the implicit function theorem gives the desired neighborhoods U and V and a C^1 inverse F^{-1}. The estimate (4.95) follows immediately from (4.93).

If u is real analytic, then so is G. It follows immediately from (4.97) that F is real analytic away from 0. Since ϕ is the composition of a real analytic function with F it is real analytic where F is (namely, away from 0). ∎

The image minimal surface is locally a multi-valued graph over the plane $x_3 = 0$. We will next analyze the height function ϕ. Taking into account the parameterization (4.94), we can write the minimal surface equation for ϕ from Lemma 4.31 as

$$\operatorname{div}\left(\frac{\nabla\phi}{W}\right) = 0\,, \tag{4.98}$$

where

$$W = \left(1 + \frac{|\nabla\phi|^2}{d^2|z|^{2d-2}}\right)^{\frac{1}{2}}\,. \tag{4.99}$$

When $d = 1$, (4.98) is the classical minimal surface equation (described in Chapter 1). For $d \ge 2$, (4.95) implies that W is bounded so that (4.98) is again a uniformly elliptic equation.

Lemma 4.32 *Suppose that $d \geq 1$ is fixed, ϕ_1 and ϕ_2 are solutions on D to* (4.98) *with W as in* (4.99)*, $\phi_i(0) = 0$, and $|\phi_i(z)| \leq C_0\,|z|^{d+1}$. If $\Phi = \phi_1 - \phi_2$ is not identically zero, then there exists an integer $n \geq d+1$ (in fact, $n \geq d+2$) and an asymptotic expansion*

$$\Phi(z) = \mathrm{Re}\; c\, z^n + \rho(z)\,, \tag{4.100}$$

where $c \neq 0$ and

$$|\rho(z)| + |z|\,|\nabla \rho(z)| \leq C|z|^{n+\epsilon} \tag{4.101}$$

for some $\epsilon > 0$ and $C < \infty$.

PROOF: As in the proof of the strong maximum principle (Lemma 1.17), we conclude that Φ satisfies an equation of the form

$$0 = (a_{i,j}\,\Phi_{x_j})_{x_i} + b_i\,\Phi_{x_i} + c\,\Phi\,, \tag{4.102}$$

where a_{ij}, b_i, c are smooth, a_{ij} is symmetric, uniformly elliptic, and $a_{ij}(0) = \delta_{ij}$.

We are now in a position to apply Theorem 4.27 to Φ. We conclude that if Φ does not vanish identically, then there is an asymptotic expansion of the form (4.100) since $\Phi(0) = 0$. ∎

The cases $d \geq 2$ of Lemma 4.32 will be applied to study branch points of minimal surfaces. In later sections, we will apply Lemma 4.32 to the minimal surface equation (that is, the case $d = 1$) when we analyze the intersections of immersed minimal surfaces.

Suppose now that ζ is a d-th root of unity (that is, $\zeta^d = 1$). Then $\phi_\zeta(z) \equiv \phi(\zeta\, z)$ also satisfies (4.98). Moreover, the self-intersections of the minimal surface are the points where ϕ_ζ and ϕ agree (as we allow all possible values of ζ). Fixing ζ for now, define the function Φ by

$$\Phi(z) = \phi_\zeta(z) - \phi(z)\,. \tag{4.103}$$

By Lemma 4.32, if Φ is not identically zero, we get an asymptotic expansion of the form

$$\Phi(z) = \mathrm{Re}\; c\, z^n + \rho(z)\,, \tag{4.104}$$

where $c \neq 0$ and

$$|\rho(z)| + |z|\,|\nabla \rho(z)| \leq C|z|^{n+\epsilon} \tag{4.105}$$

for some $\epsilon > 0$.

Arguing exactly as in Lemma 4.31, we can make a C^1 change of coordinates F which is conformal at the origin such that on some ball B_{r_0} we can represent Φ as

$$\Phi(z) = \mathrm{Re}\, F(z)^n\,. \tag{4.106}$$

Finally, the representation (4.106) allows us to give a good description of u near a branch point.

Theorem 4.33 *Let $u : D \to \mathbb{R}^3$ be an almost conformal harmonic map (so that the image is minimal). There exists some neighborhood V containing the origin such that either $u(V)$ is an immersed surface or there exist simple C^1 arcs $\gamma_i : [0,1] \to D$ with $\gamma_i(0) = 0$, $|\gamma_i'(0)| = 1$, $\gamma_1'(0) \neq \gamma_2'(0)$, $u(\gamma_1(t)) = u(\gamma_2(t))$ for all t, and such that the tangent spaces to the image of u meet transversely along the images of the γ_i.*

PROOF: We may suppose that u has a branch point at 0 (otherwise the first option holds trivially). By Lemma 4.31, there is a neighborhood V such that, after a coordinate change, we have the representation $u_1 + i\,u_2 = z^n$ and $u_3(z) = \phi(z)$ where (4.95) holds. Let $\zeta = \mathrm{e}^{2\pi i/n}$ and $\Phi = \phi_\zeta - \phi$.

If Φ is identically zero, then $\phi(z) = \phi(\zeta\, z)$ for all z and hence $\phi(z) = \bar{\phi}(z^n)$ for a smooth function $\bar{\phi}$. We can then reparametrize the image $u(V)$ by taking $\bar{u}(z) = u(z^{1/n})$. The image of $\bar{u}$ is an immersed surface and it clearly coincides with the image of u.

Otherwise, if Φ does not vanish identically, then we have the representation (4.106) for Φ in the coordinates given by F. In these coordinates,

$$\Phi(t\,\mathrm{e}^{\frac{\pi i}{2n}}) = \mathrm{Re}(i\,t^n) = 0 \tag{4.107}$$

for $0 \leq t \leq r_0$. Since, by definition, Φ vanishes when ϕ and ϕ_ζ agree, (4.107) implies that

$$\phi(t\,\mathrm{e}^{\frac{\pi i}{2n}}) = \phi_\zeta(t\,\mathrm{e}^{\frac{\pi i}{2n}}) = \phi(t\,\mathrm{e}^{\frac{5\pi i}{2n}}) \tag{4.108}$$

for $0 \leq t \leq r_0$. Therefore, in the coordinates given by F, the curves γ_i are given by the rays from the origin in B_{r_0} with angles $\frac{\pi}{2n}$ and $\frac{5\pi}{2n}$.

Finally, since $\{|\nabla\,\mathrm{Re}\,z^n| = 0\} = \{0\}$, the representation (4.106) implies that $\nabla\Phi$ does not vanish along the γ_i (except at the origin). Consequently, the tangent spaces to the image of u meet transversely along the images of the γ_i. ∎

Definition 4.34 If 0 is a branch point of u but there exists some neighborhood V containing the origin such that $u(V)$ is an immersed surface, then we say that 0 is a *false branch point*. If there exists some neighborhood V containing the origin and simple C^1 arcs $\gamma_i : [0,1] \to D$ with $\gamma_i(0) = 0$, $|\gamma_i'(0)| = 1$, $\gamma_1'(0) \neq \gamma_2'(0)$, $u(\gamma_1(t)) = u(\gamma_2(t))$ for all t, and such that the tangent spaces to the image of u meet transversely along the images of the γ_i, then 0 is a *true branch point*.

False branch points come from the parameterization whereas true branch points are visible in the geometry of the image surface. For this reason, it is significantly easier to rule out true branch points for area-minimizing disks. We will return to this in the next section.

4.7 The Absence of True Branch Points

In this section, we will prove that a least area map $u : D \to \mathbb{R}^3$ does not have any true branch points; in other words, the image of u must be an immersed surface.

In particular, if u is a solution to the Plateau problem (see Definition 4.13), then $u(D)$ is an immersed surface. True branch points were ruled out by R. Osserman in [Os1]. This leaves open the possibility that u has false branch points (where the problem is with the parameterization and not the surface). This will be addressed in the next section.

Theorem 4.35 (Osserman [Os1]) *If $u : D \to \mathbb{R}^3$ is an almost conformal harmonic map with a true branch point at the origin, then there exists a map $v : D \to \mathbb{R}^3$ such that u and v have the same image, u and v are homeomorphic on ∂D, and v is not stationary for the energy functional with respect to some compactly supported smooth variation.*

As an immediate consequence, we see that if u has a true branch point it cannot be area-minimizing.

Corollary 4.36 (Osserman [Os1]) *If $u : D \to \mathbb{R}^3$ is a solution to the Plateau problem, then it cannot have any true branch points.*

PROOF: If u had a true branch point, then Theorem 4.35 would give a new map with the same energy which is not minimizing. This gives a contradiction. ∎

In the proof of Theorem 4.35, we will use the existence of a true branch point to construct a map (via cut-and-paste arguments) with the same energy which now folds along a curve. Since stationary maps from the disk have isolated singularities, the new map cannot be stationary.

PROOF OF THEOREM 4.35 As promised we will construct an almost conformal map v with the same image as u so that

$$E_v = 2 \text{ Area}_v = 2 \text{ Area}_u = E_u \tag{4.109}$$

and such that v is not stationary for the energy. Before doing this, we will construct a map $\bar{v}$ with the same image as u that is continuous and is an immersion away from a set of measure zero. Morrey's version of the uniformization theorem guarantees that we can reparametrize $\bar{v}$ to get the almost conformal map v.

Suppose that 0 is a true branch point of $u : D \to \mathbb{R}^3$ and let γ_i, $i = 1, 2$, and V be given by Theorem 4.33. We may suppose that ∂V is smooth and that each curve γ_i intersects ∂V transversally. In fact, after reparameterizing the γ_i, we can arrange that

$$\gamma_i \cap \partial V = \gamma_i(\epsilon) \tag{4.110}$$

for some $\epsilon > 0$.

Choose a homeomorphism $\bar{F} : \overline{D_\epsilon} \to \overline{V}$ which is C^2 away from the origin with $\bar{F}(it) = \gamma_1(t)$ and $\bar{F}(-it) = \gamma_2(t)$ for $0 \leq t \leq \epsilon$.

We will next construct a map $G : \overline{D_\epsilon} \to \overline{D_\epsilon}$ with the following properties. First, G is the identity on ∂D_ϵ and continuous in a neighborhood of ∂D_ϵ. Second, for $-\frac{\epsilon}{2} \leq t \leq \frac{\epsilon}{2}$

$$\lim_{s \to 0_+} G(t + is) = 2\, i\, |t| \tag{4.111}$$

and

$$\lim_{s \to 0_-} G(t + is) = -2\, i\, |t| \, . \tag{4.112}$$

Third, even though G will not be continuous across the real axis, $u(\bar{F}(G))$ is will be continuous everywhere. This is possible since (4.111) and (4.112) imply that $u(\bar{F}(G(t))) = u(\gamma_i(|t|))$. Fourth, $u(\bar{F}(G))$ is piecewise C^2 and an immersion almost everywhere.

There is clearly a great deal of freedom in constructing the map G. We describe one possible construction. Choose a discontinuous map $G : \overline{D_\epsilon} \to \overline{D_\epsilon}$ such that the negative and positive parts of the imaginary axis are mapped to $-i\epsilon$ and $i\epsilon$, respectively, $-\frac{\epsilon}{2}$ and $\frac{\epsilon}{2}$ are taken to the origin, the segments of discontinuity $[-\frac{\epsilon}{2}, 0]$ and $[0, \frac{\epsilon}{2}]$ are mapped according to (4.111) and (4.112), G is the identity on ∂D_ϵ and continuous on a neighborhood of ∂D_ϵ, and G is a diffeomorphism on each connected component of $D_\epsilon \setminus \big([-\frac{\epsilon}{2}, 0] \cup [0, \frac{\epsilon}{2}] \cup [-i\epsilon, i\epsilon]\big)$. The map G was constructed so that $u(\bar{F}(G))$ is continuous and piecewise C^2.

Define a new map $\bar{v}$ by

$$\begin{aligned} \bar{v}(z) &= u(\bar{F}(G(\bar{F}^{-1}(z)))) \text{ for } z \in V \text{ and} \\ \bar{v}(z) &= u(z) \text{ for } z \notin V \, . \end{aligned} \tag{4.113}$$

The map $\bar{v}$ is continuous, piecewise C^2 and is an immersion away from a set of measure zero.

Morrey's version of the uniformization theorem (see Remark 4.3) implies that there is homeomorphism $T : D \to D_\epsilon$ in W^2 (so that its Hessian is in L^2) such that $v = \bar{v} \circ T$ is almost conformal. By construction, the images of u and v are identical so that they have equal area.

However, the transversality of the image minimal surface along the γ_i implies that the map v now has branch points along two entire segments. Therefore v cannot be stationary for energy since, by Corollary 4.15, the branch points for a stationary map are isolated. ∎

4.8 The Absence of False Branch Points

The main result of this section, Theorem 4.38, shows that a solution to the Plateau problem $u : D \to \Omega \subset \mathbb{R}^3$, where Ω is mean convex and $u(\partial D)$ is embedded, does not have any false branch points. Combined with Osserman's result from the previous section, it follows that the energy-minimizing map u is an immersion.

The most general version of this result is due to R. Gulliver, theorem 8.2 of [Gu] (see also H. W. Alt [Alt]). Recall that u is said to be a solution to the Plateau problem with boundary Γ if u minimizes energy among all maps whose restriction to ∂D is a monotone map to Γ (see Definition 4.13).

Theorem 4.37 (Absence of False Branch Points, [Gu]; cf. [Alt]) *Let M^3 be a three-dimensional C^3 Riemannian manifold and $\Gamma \subset M$ a piecewise C^1 Jordan curve. If $u : D \to \mathbb{R}^3$ is a solution of the Plateau problem with boundary Γ, then u is an immersion on D. That is, u has no interior branch points.*

Since our primary interest will be in the special case where the boundary of $u(D)$ lies in a mean convex domain, we will not further discuss Theorem 4.37. The question of boundary branch points is not completely solved except in the case of real analytic boundary by Gulliver and F. Leslie [GuLe]. On the other hand, the definitive boundary regularity result was proven for minimizing currents by Hardt and Simon in [HaSi] (see [Wh2] for the latest developments in the study of boundary regularity for minimal surfaces).

The main theorem of this section is the following:

Theorem 4.38 (Absence of False Branch Points) *Let Ω be a bounded mean convex region with smooth boundary and suppose that $u : \bar{D} \to \bar{\Omega} \subset \mathbb{R}^3$ is a solution of the Plateau problem as in Definition 4.13. If $u(\partial D) \subset \partial\Omega$ is embedded and $u(D) \cap \Omega \neq \emptyset$, then $u(D) \subset \Omega$ and u has no false branch points.*

Let $u : D \to \Omega \subset \mathbb{R}^3$ be a solution to the Plateau problem so that u is almost conformal, harmonic, and $u : \partial D \to \partial\Omega$ is monotone. In Corollary 4.36 (in the previous section), we saw that the image $u(D)$ is an immersed surface.

By Lemma 4.31, after a C^1 change of coordinates which is real analytic away from 0, there exists a neighborhood V of $0 \in D$ and an integer $d < \infty$ such that for $z \in V$

$$u_1 + iu_2 = z^d \tag{4.114}$$

and $u_3 = \phi(z)$, where

$$|\phi(z)| + |z|\,|\nabla\phi(z)| + |z|^2\,|\nabla^2\phi(z)| \leq C|z|^{d+1}\,. \tag{4.115}$$

If $d \geq 2$, then 0 is a branch point for u. In this case, 0 is a false branch point if

$$\phi(z) \equiv \phi(\zeta_d\, z) \tag{4.116}$$

for all $z \in V$ and for any d-th root of unity ζ_d. By the results of the previous section, all interior branch points are false.

Of course, if 0 is not a branch point, then we have (4.114) and (4.115) with $d = 1$.

Before proving the theorem, we will need some preliminary lemmas.

Lemma 4.39 *Given any $z \in \bar{D}$ and compact set $K \subset D$, the set $K \cap u^{-1}(u(z))$ is finite.*

PROOF: For any point $z_j \in K \cap u^{-1}(u(z)) \subset D$, the representation (4.114) implies there is a neighborhood V_{z_j} of z_j such that

$$V_{z_j} \cap u^{-1}(u(z)) = \{z_j\}\,. \tag{4.117}$$

Since u is continuous on $\bar{D}$, the set $u^{-1}(u(z))$ is compact and so is $K \cap u^{-1}(u(z))$. Therefore, there is a finite set of distinct points $z_1, \ldots, z_k$ such that

$$K \cap u^{-1}(u(z)) \subset \bigcup_{j=1}^{k} V_{z_j}\,. \tag{4.118}$$

Combining (4.117) and (4.118), we have $K \cap u^{-1}(u(z)) = \{z_j\}_{j=1,\dots,k}$ and the lemma follows. ∎

We will now specialize to the case where the boundary of the minimal surface is contained in the boundary of a mean convex region. This situation will be the focus of the next section.

If the region is actually convex, then things are simpler, as the following example illustrates:

Example 4.40 Suppose that $u : \bar{D} \to \bar{B}_1 \subset \mathbb{R}^3$ is a continuous (on $\bar{D}$) harmonic map (not necessarily almost conformal) with $u(\partial D) \subset \partial B_1$ and $u(D) \cap B_1 \neq \emptyset$. Define $v : \bar{D} \to \mathbb{R}$ by $v(z) = 1 - |u(z)|^2$. Since $v \geq 0$, $v(\partial D) = 0$, and $\Delta v \leq 0$, the strong maximum principle implies that v is positive in D. In other words, $u(D) \subset B_1$. Furthermore, the Hopf boundary point lemma implies that the normal derivative to v does not vanish on ∂D. If u is, in addition, almost conformal and C^1 on $\bar{D}$, this nonvanishing implies that there are no branch points in a neighborhood of ∂D.

In the next lemma (Lemma 4.41), we will generalize this example to the case where $u : \bar{D} \to \bar{\Omega} \subset \mathbb{R}^3$ is an almost conformal harmonic map with $u(\partial D) \subset \partial\Omega$ and $\partial\Omega$ is mean convex. To do this we will need to compute the Hessian of the distance function to the boundary of a mean convex region.

Let $\Omega \subset \mathbb{R}^3$ be a mean convex region in $\mathbb{R}^3$ and let ρ denote the distance to the boundary, that is,

$$\rho(x) = \operatorname{dist}(\partial\Omega, x)\,. \tag{4.119}$$

For $y \in \partial\Omega$, let $\kappa_1(y) \leq \kappa_2(y)$ denote the principal curvatures of $\partial\Omega$. Since the boundary $\partial\Omega$ is mean convex, we have

$$0 \leq \min_{y\in\partial\Omega} [\kappa_1(y) + \kappa_2(y)]\,. \tag{4.120}$$

Since $\partial\Omega$ is smooth and compact,

$$\max_{y\in\partial\Omega} [|\kappa_1(y)| + |\kappa_2(y)|] \leq \bar{\kappa} < \infty\,. \tag{4.121}$$

Combining (4.120) and (4.121), for any $y \in \partial\Omega$ and $t < 1/\bar{\kappa}$, we get

$$\begin{aligned} 0 &\leq \frac{\kappa_1(y) + \kappa_2(y) - 2\,t\,\kappa_1(y)\,\kappa_2(y)}{(1-\kappa_1(y)\,t)\,(1-\kappa_2(y)\,t)} \\ &= \frac{\kappa_1(y)}{1-\kappa_1(y)\,t} + \frac{\kappa_2(y)}{1-\kappa_2(y)\,t}\,. \end{aligned} \tag{4.122}$$

If $\kappa_1(y) \leq 0$, then (4.122) is immediate. If $0 < \kappa_1(y)$, we use the fact that $t < 1/\bar{\kappa}$ and hence

$$2\,t\,\kappa_1(y)\,\kappa_2(y) \leq t\,(\kappa_1^2(y) + \kappa_2^2(y)) < \kappa_1(y) + \kappa_2(y)\,. \tag{4.123}$$

By (4.121), there is a tubular neighborhood $\Omega_\delta = \{\rho < \delta\} \cap \Omega$ such that each $x \in \Omega_\delta$ has a unique closest point $x' \in \partial\Omega$. In other words, the normal exponential map is a diffeomorphism onto Ω_δ. We can therefore use the Riccati equation to compute the Hessian of the distance function to $\partial\Omega$ on Ω_δ (this is done on page 355 of [GiTr]). Given $x \in \Omega$ with $\text{dist}(\partial\Omega, x) = t < \delta$, then $|\nabla \rho| = 1$ ($\nabla\rho$ is the unit normal to the level set $\{\rho = t\}$). If $x' \in \partial\Omega$ is the closest point to x, then the eigenvalues of the Hessian of ρ at x are given by

$$\frac{-\kappa_1(x')}{1-\kappa_1(x')\,t}\,, \quad \frac{-\kappa_2(x')}{1-\kappa_2(x')\,t}\,, \quad 0\,. \tag{4.124}$$

Given any C^2 function $f : \mathbb{R} \to \mathbb{R}$, the Hessian of $f \circ \rho$ is

$$(f(\rho))_{x_i\,x_j} = f''(\rho)\,\rho_{x_i}\,\rho_{x_j} + f'(\rho)\,\rho_{x_i\,x_j}\,. \tag{4.125}$$

If $f(t) = 2\,a\,t - t^2$ for some constant $a > 0$, so that $f \circ \rho = 2\,a\,\rho - \rho^2$, then $f'(t) = 2\,a - 2\,t$ and $f''(t) = -2$. Combining (4.124) and (4.125), if $x' \in \partial\Omega$ is the closest point to x, then the eigenvalues of the Hessian of $2\,a\,\rho - \rho^2$ at x are given by

$$-2(a-t)\,\frac{\kappa_1(x')}{1-\kappa_1(x')\,t}\,, \quad -2(a-t)\,\frac{\kappa_2(x')}{1-\kappa_2(x')\,t}\,, \quad -2\,. \tag{4.126}$$

Lemma 4.41 *Let Ω be a bounded mean convex region with smooth boundary, and suppose that $u : \bar{D} \to \bar{\Omega} \subset \mathbb{R}^3$ is continuous, almost conformal, and harmonic. If $u(\partial D) \subset \partial\Omega$ and $u(D) \cap \Omega \neq \emptyset$, then $u(D) \subset \Omega$ and $|\nabla u|$ does not vanish on ∂D.*

Notice that this implies that $u(\bar{D})$ intersects the boundary transversely.

PROOF: Choose $a > 0$ small enough so that

$$a\,\bar{\kappa} < \frac{1}{2}\,; \tag{4.127}$$

then for $t < \delta_1 = \min\{\delta, \frac{1}{2\,\bar{\kappa}}\}$ we have

$$(a-t)\,\frac{\bar{\kappa}}{1-\bar{\kappa}\,t} < 1\,. \tag{4.128}$$

Set

$$f(t) = 2\,a\,t - t^2 \tag{4.129}$$

and

$$v(z) = f \circ \rho(u(z))\,. \tag{4.130}$$

Combining (4.126) and (4.128), we get that the eigenvalues of $\text{Hess}_{f\circ\rho}(\bar{y})$ for $\rho(y) \leq \delta_1$ are

$$-2 < -2(a-t)\,\frac{\kappa_2(y)}{1-\kappa_2(y)\,t} \leq -2(a-t)\,\frac{\kappa_1(y)}{1-\kappa_1(y)\,t}\,, \tag{4.131}$$

where $y \in \partial\Omega$ is the closest point to $\bar{y}$.

We will show that $v(z)$ is superharmonic if $u(z)$ is in some tubular neighborhood of $\partial\Omega$. Using this, it will again follow that $u(D) \subset \Omega$ and that the normal derivative of u does not vanish at the boundary.

Let $z = x_1 + i\,x_2$ be coordinates for D and y_1, y_2, y_3 be coordinates for $\mathbb{R}^3$. Using the chain rule and the fact that $\Delta u = 0$, we get

$$\begin{aligned}\Delta v(z) = v_{x_i\,x_i}(z) &= (f \circ \rho)_{y_j\,y_k}(u(z))\,(u_j)_{x_i}(z)\,(u_k)_{x_i}(z) \\ &\quad + (f \circ \rho)_{y_j}(u(z))\,(u_j)_{x_i\,x_i}(z) \\ &= (f \circ \rho)_{y_j\,y_k}(u(z))\,(u_j)_{x_i}(z)\,(u_k)_{x_i}(z)\,.\end{aligned} \tag{4.132}$$

If we let Π_z denote the two-plane spanned by $u_{x_1}(z)$ and $u_{x_2}(z)$, then (4.132) together with the fact that u is almost conformal gives

$$\begin{aligned}\Delta v(z) &= \mathrm{Hess}_{f\circ\rho}(u(z))\,(u_{x_i}(z), u_{x_i}(z)) \\ &= |u_{x_1}(z)|^2\ \mathrm{Tr}\,\mathrm{Hess}_{f\circ\rho}(u(z))\big|_{\Pi_z}\,.\end{aligned} \tag{4.133}$$

Suppose now that $\rho(u(z)) = t \leq \delta_1$ so that (4.131) applies and hence

$$\begin{aligned}&\max_{\text{two planes }\Pi}\ \mathrm{Tr}\,\mathrm{Hess}_{f\circ\rho}(u(z))\big|_{\Pi} \\ &\qquad = -2(a-t)\left(\frac{\kappa_1(y)}{1-\kappa_1(y)\,t} + \frac{\kappa_2(y)}{1-\kappa_2(y)\,t}\right) \leq 0\,,\end{aligned} \tag{4.134}$$

where the last inequality follows from (4.122) and y is the closest point in $\partial\Omega$ to $u(z)$.

Combining (4.133) and (4.134), v is superharmonic so long as $\rho(u(z)) \leq \min\{\delta_1, a\}$. Since v is nonnegative, vanishes on the boundary, and $u(D) \cap \Omega \neq \emptyset$, it must be strictly positive on D by the maximum principle. By the definition of v this implies that $u(D) \subset \Omega$. Finally, since v achieves its minimum on every point of the boundary, the Hopf boundary point lemma implies that the normal derivative $\frac{dv}{dn}$ is strictly negative on the boundary. Combining this with the chain rule, we get that in the weak sense

$$0 < \frac{dv}{dn} = (f \circ \rho)_{y_j}(u)\,\frac{du_j}{dn} = 2\,a\left\langle \nabla\rho(u)\,,\,\frac{du}{dn}\right\rangle\,, \tag{4.135}$$

and hence ∇u does not vanish on the boundary. ∎

As a consequence, we see that the restriction of u to the boundary is an embedding and u can have at most finitely many interior branch points.

Corollary 4.42 *Let Ω be a bounded mean convex region with $\partial\Omega$ smooth, and suppose that $u : \bar{D} \to \bar{\Omega} \subset \mathbb{R}^3$ is C^1 on $\bar{D}$, almost conformal, and harmonic. Let $\mathcal{B}_0$ denote the branch points of u and define $\mathcal{B} = u^{-1}(u(\mathcal{B}_0))$. If $u(\partial D) \subset \partial\Omega$ and $u(D) \cap \Omega \neq \emptyset$, then $\mathcal{B}$ is finite and there is a compact set $K \subset D$ such that $\mathcal{B} \subset K$. In particular, the restriction of u to ∂D is an embedding.*

PROOF: The previous lemma (see (4.135)) gives that $|\nabla u| > 0$ on ∂D and hence, since u is almost conformal, there are no branch points on the boundary of D. Since $u \in C^1(\bar{D})$, this implies that ∇u does not vanish on a neighborhood of the boundary (so that there are no branch points in a neighborhood of the boundary). Therefore, we have that the branch points are compactly contained in D. Since the local description (4.114) implies that they are isolated there can be only finitely many. Let $\mathcal{B}_0 = \{z_1, \dots, z_k\}$ denote the branch points and let $\mathcal{B} = u^{-1}(u(\{z_1, \dots, z_k\}))$.

The previous lemma implies that $u^{-1}(u(z_j)) \subset D$ for each j. Since u is continuous, this implies that $u^{-1}(u(z_j))$ is compactly contained in D. Therefore, Lemma 4.39 implies that $u^{-1}(u(z_j))$ is a finite set. Doing this for each $j = 1, \dots, k$, we get that $\mathcal{B}$ is a finite set of points which is compactly contained in D. ∎

We are now prepared to prove the main theorem of this section.

PROOF OF THEOREM 4.38 We will suppose that 0 is a false branch point for u of order $d \geq 2$ and that $u(D)$ is not contained in a plane and deduce a contradiction.

We begin by showing that there is some plane Π through $u(0)$ such that Π intersects $u(D)$ transversely and

$$\Pi \cap u(\mathcal{B}) = \{u(0)\}\,. \tag{4.136}$$

Let $\tilde{\Pi}$ denote the set of two planes in $\mathbb{R}^3$ through $u(0)$ such that (4.136) holds.

Since $u(\mathcal{B})$ is a finite set of points, there is some finite set of lines through $u(0)$ which intersect $u(\mathcal{B}) \setminus \{u(0)\}$. Each of these lines is contained in a one-parameter family of planes through $u(0)$ so that $\tilde{\Pi}$ is an open set with full measure.

Define the map $P : \mathbf{S}^1 \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\begin{aligned} P(\theta_1, \theta_2, (a_1, a_2)) = u(0) &+ a_1\,(-\sin\theta_1, \cos\theta_1, 0) \\ &+ a_2\,(\cos\theta_1\ \cos\theta_2, \sin\theta_1\ \cos\theta_2, -\sin\theta_2) \end{aligned} \tag{4.137}$$

so that for each (θ_1, θ_2) we get an affine map whose image is a two-plane through $u(0)$. Since the differential of P is surjective at every point (as a map to $\mathbb{R}^3$), the parametric version of Sard's theorem applies. Consequently, for any compact immersed submanifold $N \subset \mathbb{R}^3$ the set of (θ_1, θ_2) for which

$$P(\theta_1, \theta_2, \mathbb{R}^2) \tag{4.138}$$

intersects N transversely is of full measure. In particular, we can find a plane $\Pi \in \tilde{\Pi}$ which intersects $u(D)$ transversely.

After a rotation of $\mathbb{R}^3$ fixing $u(0)$ and a translation, we may suppose that $u(0) = 0$ and that Π is given by $y_1 = 0$. The fact that Π and $u(D)$ intersect transversely implies that

$$\nabla_{u(D)} y_1 \neq 0\,, \tag{4.139}$$

and hence the nodal set $\{y_1 = 0\} \cap u(D)$ is a collection of compact immersed arcs. Let $\tilde{\gamma}$ denote the connected component of $\{y_1 = 0\} \cap u(D)$ which contains $u(0)$.

We can write

$$\tilde{\gamma} = \bigcup_{j=1}^{\ell} \gamma_j \,, \tag{4.140}$$

where each γ_j is a compact immersed curve which is either closed or has two endpoints. Suppose that $u(0) \in \gamma_1$ and hence $0 \in u^{-1}(\gamma_1) \subset \{u_1 = 0\}$.

Moreover, (4.136) and (4.139) imply that $u^{-1}(\gamma_1)$ is a connected curve which is smooth away from 0. Since 0 is a branch point of order $d \geq 2$, we have the description (4.114) for u. Therefore, in a neighborhood of 0, $u^{-1}(\gamma_1)$ is the union of $2d$ disjoint arcs.

Next, we claim that these $2d$ arcs can never intersect again. If any two did, then we would have a bounded nodal domain for the harmonic function u_1. In that case, u_1 would have to be identically zero by the maximum principle implying that $u(D)$ would be contained in the plane $\{y_1 = 0\}$. Therefore these $2d$ arcs stay disjoint to the boundary. Each of these arcs begins at 0 and ends in ∂D (here we used Lemma 4.41 again). The $2d$ endpoints of these curves in ∂D are mapped by u into the 2 endpoints of γ_1. We conclude that two distinct arcs α_1 and α_2 satisfy

$$u(\alpha_i(0)) = u(0) \text{ and } u(\alpha_1(1)) = u(\alpha_2(1)) \,, \tag{4.141}$$

where $u(\alpha_1(1)) \in \partial u(D)$. Let σ denote the portion of ∂D between $\alpha_1(1)$ and $\alpha_2(1)$. Since the map u is monotone on the boundary and the curve $u(\partial D)$ is embedded, $u^{-1}(u(\alpha_1(1)))$ is connected and hence $u(\sigma) = u(\alpha_1(1))$. Therefore, the maximum principle implies that u_1 vanishes on the domain bounded by α_1, α_2, and σ. By unique continuation, u_1 vanishes identically on D, giving the desired contradiction. ∎

4.9 Embedded Solutions of the Plateau Problem

In this section, we will see that in certain cases the solution to the Plateau problem must be embedded. Following Meeks and Yau, we will see that if the boundary curve is embedded and lies on the boundary of a smooth convex set (and it is null-homotopic in this convex set), then the minimizing solution is embedded. In the previous section, we saw that the solution had to be immersed in this case. Note that some restriction on the boundary curve is certainly necessary to prove such an embeddedness theorem. For instance, if the boundary curve was knotted (for instance, the trefoil), then it could not be spanned by any embedded disk (minimal or otherwise).

Theorem 4.43 (Meeks-Yau [MeY1]) *Let M^3 be a compact Riemannian three-manifold whose boundary is mean convex and let γ be a simple closed curve in ∂M which is null-homotopic in M; then γ is bounded by a least area disk and any such least area disk is properly embedded.*

Theorem 4.43 is contained in their results (see [MeY1] and [MeY2] and references therein for additional results).

We will consider the special case of the above theorem where M is a compact mean convex region in $\mathbb{R}^3$.

Theorem 4.44 (Meeks-Yau [MeY1]) *Let Ω be a bounded mean convex region in $\mathbb{R}^3$ with smooth boundary and $\Gamma \subset \partial\Omega$ a C^1 simple closed curve. If Γ is null homotopic in Ω, then the solution $u : D \to \Omega$ of the Plateau problem is a proper embedding.*

Note that Rado's theorem, Theorem 4.29, can be viewed as a special case of Theorem 4.44. In this case, the boundary curve is on the boundary of a convex cylinder. In general, a curve is said to be extremal if it lies on the boundary of its convex hull. Prior to the work of Meeks and Yau, embeddedness was known for extremal boundary curves in $\mathbb{R}^3$ with small total curvature by the work of Gulliver and J. Spruck [GuSp]. Subsequently, Almgren and Simon [AmSi] and Tomi and A. J. Tromba [ToTr] proved the existence of some embedded solution for extremal boundary curves in $\mathbb{R}^3$ (but not necessarily for the Douglas-Rado solution produced in Section 4.3).

By the results of the previous sections, there exists a solution u to the Plateau problem which is a smooth proper immersion such that its restriction to ∂D is an embedding. For the remainder of this section, u will always be assumed to have these properties.

The following lemma shows that u is an embedding in a neighborhood of the boundary:

Lemma 4.45 *With u as above there exists some $\delta, \bar{\delta} > 0$ such that $u : D \setminus D_{1-\delta} \to \Omega$ is an embedding and $u(D)$ is embedded in a $\bar{\delta}$ neighborhood of $\partial\Omega$.*

PROOF: We will show that $u(w_1) \neq u(w_2)$ for any w_1, w_2 in $D \setminus D_{1-\delta}$ so long as δ is sufficiently small.

Since u is an immersion and is C^1 on $\bar{D}$, there is an $\epsilon_0 > 0$ such that

$$u(D_{\epsilon_0}(z_0) \cap D) \tag{4.142}$$

is an embedded surface for any $z_0 \in D$. Consequently, we may assume that $|w_1 - w_2| > \epsilon_0$.

Since u is C^1 on $\bar{D}$, there is a constant C_0 such that for any $z_1, z_2 \in \bar{D}$

$$|u(z_1) - u(z_2)| \leq C_0 \, |z_1 - z_2| \, . \tag{4.143}$$

The fact that u restricted to ∂D is a C^1 embedding implies that there is some constant $C_1 > 0$ such that for any $z_1, z_2 \in \partial D$

$$C_1 \, |z_1 - z_2| \leq |u(z_1) - u(z_2)| \, . \tag{4.144}$$

Choose $z_1, z_2 \in \partial D$ with $|w_i - z_i| \leq \delta$. By the triangle inequality together with (4.143) and (4.144),

$$\begin{aligned} |u(w_1) - u(w_2)| &\geq |u(z_1) - u(z_2)| - 2\, C_0 \, \delta \geq C_1 \, |z_1 - z_2| - 2\, C_0 \, \delta \\ &\geq C_1 \, (\epsilon_0 - 2\delta) - 2\, C_0 \, \delta \, . \end{aligned} \tag{4.145}$$

Equation (4.145) implies that the restriction of u to $D \setminus D_{1-\delta}$ is an embedding so long as $\delta < C_1\,\epsilon_0/(2(C_0+C_1))$.

By Lemma 4.41 of the previous section, if $v = f \circ \rho \circ u$ where $f(t) = 2\,a\,t - t^2$ for some $a > 0$ and ρ is the distance to $\partial\Omega$, then v is superharmonic if $v(z) \leq \delta_0$ for some $\delta_0 > 0$ and there exists $\beta > 0$ such that for all $z \in \partial D$

$$\frac{\partial v}{\partial n}(z) > \beta\,. \tag{4.146}$$

Since u is C^1 on $\bar{D}$, there exists $\delta_3 > 0$ so that (4.146) holds if $|z| > 1 - \delta_3$. Therefore, there exist $\delta_1, \delta_2 > 0$ and $U \subset D$ so that $D_{1-\delta_1} \subset U$ and $v(\partial U) = \delta_2 < \delta_0$. As in the previous section, the maximum principle implies that $v(U) \geq \delta_2$. Therefore, there is some $\bar{\delta} > 0$ such that

$$u^{-1}(\{x \in \Omega \mid \text{dist}\,(x, \partial\Omega) < \bar{\delta}\}) \subset D \setminus D_{1-\delta}\,. \tag{4.147}$$

The second claim now follows from the first. ∎

Recall that for a point $x \in \Omega$ the density of $u(D)$ at x is given by

$$\Theta_x = \lim_{s \to 0} \frac{\text{Vol}(B_s(x) \cap u(D))}{\pi\, s^2}\,. \tag{4.148}$$

If $x \in u(D)$, then (since $u(D)$ is an immersed submanifold) in any sufficiently small ball, $B_\epsilon(x)$, we have that $B_\epsilon(x) \cap u(D)$ consists of (possibly several) minimal surfaces intersecting at x. By monotonicity, as in Corollaries 1.9 and 1.10, we see that Θ_x is equal to the number of preimages of x. That is,

$$\Theta_x = \left|u^{-1}(x)\right|\,. \tag{4.149}$$

Clearly, if $x \notin u(D)$, then $\Theta_x = 0$. In either case, (4.149) holds.

Consequently, the set of double points $\mathcal{D}$ is the set of $x \in \Omega$ with

$$\Theta_x \geq 2\,. \tag{4.150}$$

By (4.149), $\mathcal{D}$ is the smallest set such that $u(D) \setminus \mathcal{D}$ is embedded. Lemma 4.45 implies that

$$\mathcal{D} \subset \{x \in \Omega \mid \text{dist}\,(x, \partial\Omega) \geq \bar{\delta}\}\,. \tag{4.151}$$

Using monotonicity, Proposition 1.8, and Lemma 4.45, (4.149) yields a quantitative form of Lemma 4.39. Namely, there exists $C_2 = C_2(\text{Vol}(u(D)), \bar{\delta}) < \infty$ such that each point in $u(D)$ has muliplicity at most C_2.

Lemma 4.46 *With u as above, the set of double points $\mathcal{D} \subset u(D)$ is compact.*

PROOF: Lemma 4.45 shows that $\mathcal{D}$ is contained in the interior of $u(D)$ so we can apply monotonicity to $u(D)$ in a neighborhood of each point of $\mathcal{D}$.

By Corollary 1.10, the density is upper semicontinuous and $\mathcal{D}$ is closed. Since it is also bounded, $\mathcal{D}$ is compact. ∎

Using the local description of a solution to an elliptic equation, we will show that the set of nontransverse self-intersections, $\tilde{\mathcal{D}} \subset \mathcal{D}$, is a finite set of points. By definition, x is in $\tilde{\mathcal{D}}$ if and only if there are points $z_1 \neq z_2$ in $u^{-1}(x)$ with neighborhoods V_1 of z_1 and V_2 of z_2 such that $u(V_1)$ and $u(V_2)$ intersect tangentially at x. Furthermore, we may suppose that each $u(V_i)$ is a graph of a function ϕ_i over a ball B_ϵ in its tangent plane at x.

We can now apply Lemma 4.32 to get an asymptotic expansion for $\Phi = \phi_1 - \phi_2$ (which is not identically zero by unique continuation). Namely, there exists an integer $n \geq 2$ and an asymptotic expansion

$$\Phi(z) = \mathrm{Re}\; c\, z^n + \rho(z)\,, \tag{4.152}$$

where $c \neq 0$ and

$$|\rho(z)| + |z|\,|\nabla \rho(z)| \leq C|z|^{n+\alpha}\,, \tag{4.153}$$

for some $\alpha > 0$. It follows immediately from a slight variation of Lemma 4.28 that $u(V_1) \cap u(V_2)$ is homeomorphic to $2n$ arcs meeting at x.

Lemma 4.47 *With u and $\mathcal{D}$ as above, let $\tilde{\mathcal{D}} \subset \mathcal{D}$ denote the set of double points with a nontransverse self-intersection; then $\tilde{\mathcal{D}}$ is a finite set of points.*

PROOF: Suppose that $x \in \tilde{\mathcal{D}}$ with $u^{-1}(x) = \{z_1, \ldots, z_k\} \subset D$. We will argue for the case in which all of the nontransverse sheets have the same tangent plane. The modifications for the general case are clear.

Since u is a C^1 immersion, there are neighborhoods V_i of z_i such that $u(V_i)$ is graphical over P_i, where P_i is the tangent plane to $u(V_i)$ at x for each $i = 1, \ldots, k$.

Since u is proper, there exists some $\epsilon_0 > 0$ such that

$$u^{-1}(B_{\epsilon_0}(x)) \subset \bigcup_{j=1}^{k} V_j\,. \tag{4.154}$$

By the definition of $\tilde{\mathcal{D}}$, we may suppose that $P_1, \ldots, P_\ell$ all agree up to orientation and there is a definite separation between $\pm P_1$ and $P_{\ell+1}, \ldots, P_k$. Therefore, there is some $\epsilon_1 > 0$ such that for any $j_0 \leq \ell < j_1$, $B_{\epsilon_1}(x) \cap u(V_{j_0})$ and $B_{\epsilon_1}(x) \cap u(V_{j_1})$ intersect transversely.

Finally, the discussion preceding the lemma gives some $\epsilon_{j_0,j_1} > 0$ for each $j_0 \neq j_1 \leq \ell$ so that $B_{\epsilon_{j_0,j_1}}(x) \cap u(V_{j_0})$ and $B_{\epsilon_{j_0,j_1}}(x) \cap u(V_{j_1})$ intersect transversely away from x. If we now let $\epsilon > 0$ be the minimum of ϵ_0, ϵ_1, and all of the ϵ_{j_0,j_1}, then

$$B_\epsilon(x) \cap \tilde{\mathcal{D}} = \{x\} \tag{4.155}$$

and hence $\tilde{\mathcal{D}}$ is an isolated set of points.

It remains to show that $\tilde{\mathcal{D}}$ is compact. To see this, suppose that $x_j \in \tilde{\mathcal{D}}$ and let $z_j^i \in u^{-1}(x_j)$ for $i = 1, 2$ be the points whose neighborhoods intersect tangentially at x_j. Since u is a C^1 immersion, there is an $\epsilon_0 > 0$ such that

$$|z_j^1 - z_j^2| \geq \epsilon_0\,. \tag{4.156}$$

By Lemma 4.45, the z_j^i are compactly contained in Ω and hence have limit points $z^1, z^2 \in D$ which also satisfy (4.156) (and hence $z_1 \neq z_2$).

Since u is C^1, the fact that $u(z_j^1) = u(z_j^2)$ and $|N(z_j^1)| = |N(z_j^1)|$ for each j implies that $u(z^1) = u(z^2)$ and $|N(z^1)| = |N(z^1)|$, and thus $u(z^1) = u(z^2) \in \tilde{\mathcal{D}}$ since $z_1 \neq z_2$. The fact that $z_j^1 \to z_1$ and u is continuous implies that $x_j \to u(z_1)$ and hence $\tilde{\mathcal{D}}$ is compact. ∎

By the previous lemma, it follows that $\mathcal{D}$ is a collection of compact immersed curves which branch at the finite set of points $\tilde{\mathcal{D}}$.

Corollary 4.48 *With u and $\mathcal{D}$ as above, we can write*

$$\mathcal{D} = \bigcup_{j=1}^{n} \eta_j \,, \tag{4.157}$$

where each η_j is a compact immersed curve.

PROOF: The implicit function theorem implies that $\mathcal{D} \setminus \tilde{\mathcal{D}}$ is a union of C^1 immersed curves. Therefore, since Lemma 4.47 gives that $\tilde{\mathcal{D}}$ is a finite set of points and Lemma 4.46 gives that $\mathcal{D}$ is compact, the corollary follows. ∎

The final preliminary that we will need is the notion of a folding curve used by Meeks and Yau.

Definition 4.49 Let $f : D_r \to \mathbb{R}^3$ be a Lipschitz map such that the restriction of f to either $x_1 \geq 0$ or $x_1 \leq 0$ is a C^1 immersion up to the boundary. If for each x_2 either of the following two possibilities occcurs, then we say that f has a *folding curve* along the x_2-axis:

First, the plane spanned by $f_{x_2}(0, x_2)$ and $\lim_{x_1 \to 0_+} f_{x_1}(x_1, x_2)$ is transverse to the plane spanned by $f_{x_2}(0, x_2)$ and $\lim_{x_1 \to 0_-} f_{x_1}(x_1, x_2)$. Second, the vector $\lim_{x_1 \to 0_-} f_{x_1}(x_1, x_2)$ is a negative multiple of $\lim_{x_1 \to 0_+} f_{x_1}(x_1, x_2)$.

The main point of folding curves is that they cannot arise in least area maps, as is shown by the following lemma:

Lemma 4.50 *If f is a Lipschitz and piecewise C^1 map from a disk D into $\mathbb{R}^3$ which has a folding curve, then there is a piecewise C^1 Lipschitz variation vector field W which is area decreasing. In particular, f cannot be area-minimizing.*

PROOF: Suppose f is least area. Then it has zero mean curvature where it is an immersion. The definition of folding curve allows us to construct an area decreasing variation vector field W along the fold.

Since f is Lipschitz on D and the restriction of f to either $\{x_1 \geq 0\}$ or $\{x_1 \leq 0\}$ is a C^1 immersion up to the boundary, f_{x_2} is well-defined everywhere. On the other hand, the angle between $\lim_{x_1 \to 0_+} f_{x_1}(x_1, x_2)$ and $\lim_{x_1 \to 0_-} -f_{x_1}(x_1, x_2)$ is less than π. Consequently, we can choose a compactly supported Lipschitz vector field W which is piecewise C^1 such that on the one hand

$$\lim_{x_1 \to 0_+} \langle W(f(x_1, x_2)), f_{x_1}(x_1, x_2) \rangle > 0 \,, \tag{4.158}$$

but on the other hand

$$\lim_{x_1 \to 0_-} \langle W(f(x_1,x_2)), -f_{x_1}(x_1,x_2)\rangle > 0\,, \tag{4.159}$$

and finally

$$\langle W(f(0,x_2)), f_{x_2}\rangle = 0\,. \tag{4.160}$$

It follows immediately that, if we take W as the variation vector field, then the first variation of area is negative. ∎

The analog of Lemma 4.50 in one dimension less is: Given a piecewise C^1 curve γ with an interior discontinuity of the tangent γ', then there is one-parameter deformation which fixes the endpoints but decreases the length. It follows immediately that a minimizing geodesic between two points in a complete Riemannian manifold cannot have such a "fold."

We will first discuss how to use the notion of folding curves to prove the embeddedness result in the special case where $u(D)$ intersects itself transversely. The proof in the general case will combine these ideas with a perturbation argument.

The next result that we need is a purely topological result for self-transverse maps from the disk. Suppose that $v : \overline{D} \to \mathbb{R}^3$ is a C^1 proper immersion (not necessarily minimal) in general position such that the restriction of v to ∂D is an embedding.

We can still define the set of double points in the obvious way (namely points with more than one preimage). Since v is in general position, the image of v intersects itself transversely along a finite union $\mathcal{D}$ of immersed curves. Let $U_1, \ldots, U_n$ denote the connected components of $D \setminus v^{-1}(\mathcal{D})$ so that the restriction of v to each U_j is an embedding.

The disk D is given by the disjoint union of the closures of the U_j together with a series of identifications of the boundaries ∂U_j. These identifications are compatible with v in the sense that v has the same value at any two points which are identified.

The point of Proposition 4.52 below is that there is another way of identifying the boundaries of the ∂U_j to construct D in this manner which is compatible with v and such that v is now a piecewise C^1 embedding which fails to be C^1 at $\mathcal{D}$. After doing this, the new map has self-intersections along $\mathcal{D}$ but does not cross itself. By doing this, we have introduced folding curves along $\mathcal{D}$. The following example in one dimension less illustrates this:

Example 4.51 Let $v : [-\pi, \pi] \to \mathbb{R}^2$ be given by

$$v(t) = (2\,t/\pi - \sin t, \cos t)\,. \tag{4.161}$$

Then the double set is given by $(0,0)$, its inverse image is $\{-\frac{\pi}{2}, \frac{\pi}{2}\}$, and we have $U_1 = (-\pi, -\frac{\pi}{2})$, $U_2 = (-\frac{\pi}{2}, \frac{\pi}{2})$, and $U_3 = (\frac{\pi}{2}, \pi)$. Initially, $[-\pi, \pi]$ is given by taking the union of the (relative) closures of the U_i and identifying $-\frac{\pi}{2} \in \overline{U_1}$ with $-\frac{\pi}{2} \in \overline{U_2}$ and $\frac{\pi}{2} \in \overline{U_2}$ with $\frac{\pi}{2} \in \overline{U_3}$.

On the other hand, we can also construct the interval $[-\pi, \pi]$ by identifying $-\frac{\pi}{2} \in \overline{U_1}$ with $\frac{\pi}{2} \in \overline{U_2}$ and $-\frac{\pi}{2} \in \overline{U_2}$ with $\frac{\pi}{2} \in \overline{U_3}$. With this identification, v is still continuous since $v(-\frac{\pi}{2}) = v(\frac{\pi}{2})$ (this is what we mean by saying that the identification is compatible with v). By construction, the image of v is the same.

Clearly v is no longer C^1 across $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. However, the image of v does not cross itself although it does still have a self-intersection at a singular point (this is what we mean by saying that v is a piecewise C^1 embedding). We can now deform v to an embedding by making an arbitrarily small C^0 deformation in a neighborhood of $(0, 0)$. This operation, which is known as "rounding off", will reduce the length of the image of v.

We shall need the following topological proposition. See Freedman [Fr] for the proof which uses the tower construction from topology. This construction goes back to Papakyriakopolous [Pa].

Proposition 4.52 *Suppose that $v : \overline{D} \to \mathbb{R}^3$ is a C^1 proper immersion in general position such that the restriction of v to ∂D is an embedding. If $\mathcal{D}$ is the set of double points as above, let $U_1, \ldots, U_n$ denote the connected components of $D \setminus v^{-1}(\mathcal{D})$. By constructing appropriate identifications compatible with v, we can form a new disk $D' = \cup_{j=1}^n \overline{U}_j$ with the following properties. The map $w : \overline{D'} \to \mathbb{R}^3$ given by $w|_{U_i} = v|_{U_i}$ is a Lipschitz continuous piecewise C^1 embedding. The maps w and v have the same image and agree on $\partial D = \partial D'$. Finally, w is arbitrarily close in C^0 to an embedding, and each component of $\partial U_i \setminus \partial D$ is a folding curve.*

The proof of Theorem 4.44 in the case where $u(D)$ is self-transverse is now clear. Namely, we use Proposition 4.52 to produce a folding curve for a map with the same area and boundary values. Lemma 4.50 implies that this new map is not area-minimizing and, hence, neither was u. This contradiction implies that u must have been an embedding in the first place.

Using ideas of Freedman [Fr] and Freedman-Hass-Scott [FrHaSc], we can obtain the general case by combining the above approach with a perturbation argument.

Lemma 4.53 *With u and $\mathcal{D}$ as above, there exist $\epsilon > 0$ and $x_1 \in \mathcal{D}$ such that $B_\epsilon(x_1) \cap \mathcal{D}$ has only transverse self-intersections.*

PROOF: This follows immediately since the nontransverse self-intersections are isolated. ∎

Before continuing, we will recall the local description of immersed minimal surfaces near points of self-intersection. This result is a direct consequence of the analytic results on nodal and critical sets of elliptic equations (similar to the arguments in Section 4.6).

Lemma 4.54 (Freedman, Hass, and Scott [FrHaSc]) *Let $U \subset D$ be a neighborhood of 0, u_1, u_2 be smooth functions on U, and suppose that $v = u_1 - u_2$ satisfies $v = c \operatorname{Re} z^d + q(x)$ where $c \neq 0$ and q satisfies* (4.74). *Let $h : U \to V$ be a C^1*

diffeomorphism where $V \subset D$ *is a neighborhood of* 0 *such that* $c \operatorname{Re} z^d(h(x)) = v(x)$ *for* $x \in U$.

Let $r > 0$ *be such that* $D_r \subset U$ *and* ψ *be a smooth function on* U *with support in* D_r *such that* $\psi(x) = 1$ *for* $|x| \leq \frac{r}{2}$. *For* $t > 0$ *set* $u_t(x) = u_1(x) + t\,\psi(x)$*; then there exists* $t_0 > 0$ *such that for* $0 < t \leq t_0$ *the graphs of* u_t *and* u_2 *meet transversely.*

Remark 4.55 Notice that since the intersection is not changed above ∂D_r, the new curves of intersection have the same endpoints as the original (singular) curves.

We are now prepared to prove the main result of this section.

PROOF OF THEOREM 4.44 Thus far, we have shown that the solution $u : D \to \Omega$ of the Plateau problem is a proper smooth immersion which is an embedding in a neighborhood of the boundary. It remains to show that the set of double points $\mathcal{D}$ is empty. We will assume that $\mathcal{D} \neq \emptyset$ and obtain a contradiction from this.

By Lemma 4.53, there exist $\epsilon_1 > 0$ and $x_1 \in \mathcal{D}$ such that $B_{\epsilon_1}(x_1) \cap \mathcal{D}$ has only transverse self-intersections. It follows from Corollary 4.48 that are only finitely many possible cut-and-paste operations (as in Proposition 4.52) to make $B_{\epsilon_1}(x_1) \cap u(D)$ embedded. By Lemma 4.50, each of these cut-and-paste operations reduces the area of $u(D)$ by at least some $\epsilon_2 > 0$ (by perturbing along the resulting folding curves).

Lemma 4.45 implies that there is some $\delta > 0$ such that $u(D_{1-\delta}) \cap \mathcal{D} = \emptyset$. Consequently, the local description, Lemma 4.54, allows us to perturb the map u in

$$D_{1-\delta} \setminus u^{-1}(B_{\epsilon_1}(x_1)) \tag{4.162}$$

by an arbitrarily small amount to produce a new map $\tilde{u}$ which is in general position. In particular, we may assume that

$$\operatorname{Area}(\tilde{u}(D)) < \operatorname{Area}(u(D)) + \frac{\epsilon_2}{2}\,. \tag{4.163}$$

By Proposition 4.52, we may cut-and-paste D to obtain a new disk D' such that the identifications are compatible with $\tilde{u}$. The piecewise smooth embedding $w : D' \to \Omega$ has the same image, and thus the same area, as $\tilde{u}$ and these maps agree on the boundary. Since these images agree, the identifications made in $u^{-1}(B_{\epsilon_1}(x_1))$ must be among those listed above. In particular, by perturbing along one of the resulting folding curves in $B_{\epsilon_1}(x_1)$ we may reduce the area of $w(D')$ by at least ϵ_2. Together with (4.163) this contradicts the minimality of $\operatorname{Area}(u(D))$, and the proof is complete. ∎

5

Minimal Surfaces in Three-Manifolds

In this chapter, the last of these notes, we discuss the theory of minimal surfaces in three-manifolds. We begin by explaining how to extend the earlier results to this case (in particular, monotonicity, the strong maximum principle, and some of the other basic estimates for minimal surfaces). Next, we prove the compactness theorem of Choi and Schoen for embedded minimal surfaces in three-manifolds with positive Ricci curvature. An important point for this compactness result is that, by results of Choi-Wang and Yang-Yau, such minimal surfaces have uniform area bounds. The next section surveys recent results of [CM3], [CM8], [CM9], and [CM6] on compactness and convergence of minimal surfaces without area bounds. Finally, in the last section, we mention an application (from [CM7]) of the ideas of [CM8] to the study of complete minimal surfaces in $\mathbb{R}^3$.

5.1 The Minimal Surface Equation in a Three-Manifold

In this section, we will first describe the modifications necessary to discuss minimal surfaces in general Riemannian three-manifolds. The local picture is very similar to that of minimal surfaces in $\mathbb{R}^3$. In fact, many regularity results for minimal surfaces in arbitrary three-manifolds can be reduced to the corresponding global result in $\mathbb{R}^3$ by means of a rescaling argument.

In the following, M^n will denote a complete n-dimensional Riemannian manifold with sectional curvature bounded by k (i.e., $|K_M| \leq k$) and injectivity radius bounded below by $i_0 > 0$. We will usually take $n = 3$.

In order to establish the necessary local results relating curvature and area for minimal surfaces, we shall need to recall some preliminary geometric facts. The standard Hessian comparison theorem implies the following:

Lemma 5.1 *For* $r < \min\{i_0, \frac{1}{\sqrt{k}}\}$ *and any vector* X *with* $|X| = 1$

$$\left| \mathrm{Hess}_r(X,X) - \frac{1}{r} \langle X - \langle X, Dr\rangle Dr, X - \langle X, Dr\rangle Dr\rangle \right| \leq \sqrt{k}\,. \tag{5.1}$$

PROOF: The Hessian of the distance function vanishes in the radial direction. By the Hessian comparison theorem, the remaining eigenvalues of the Hessian are bounded above and below by $\sqrt{k}\,\coth\sqrt{k}\,r$ and $\sqrt{k}\,\cot\sqrt{k}\,r$, respectively. Using this, the bound (5.1) follows from elementary inequalities. ∎

Let $x \in \Sigma^2 \subset M^3$ with Σ minimal. When M is $\mathbb{R}^n$, the minimality of Σ implies that

$$\Delta r^2 = 4\,. \tag{5.2}$$

This fact is the key to proving the monotonicity formula for minimal surfaces, i.e., Proposition 1.8. In the general case, using minimality and (5.1), we get that

$$\left|\Delta r^2 - 4\right| \leq 4\sqrt{k}\,r \tag{5.3}$$

for $r < \min\{i_0, \frac{1}{\sqrt{k}}\}$. Applying the coarea formula (i.e., (1.40)) and using Stokes' theorem gives

$$s\,\frac{d}{ds}\,\text{Area}(B_s \cap \Sigma) = \int_{\partial B_s \cap \Sigma} \frac{r}{|\nabla r|} \geq \frac{1}{2}\int_{B_s \cap \Sigma} \Delta r^2\,. \tag{5.4}$$

Combining (5.3) and (5.4) implies that for $s < \min\{i_0, \frac{1}{\sqrt{k}}\}$

$$\frac{d}{ds}\left(\mathrm{e}^{2\sqrt{k}\,s}\; s^{-2}\;\text{Area}(B_s \cap \Sigma)\right) \geq 0\,. \tag{5.5}$$

We could argue similarly, as in Proposition 1.11, to obtain a mean value inequality for minimal surfaces in a three-manifold.

When $M = \mathbb{R}^3$, the Gauss map of a minimal surface is conformal. More generally, the Gauss equation and minimality together imply that

$$K_\Sigma = K_M - \frac{1}{2}|A|^2\,, \tag{5.6}$$

so that

$$|A|^2 \leq 2\,k - 2\,K_\Sigma\,. \tag{5.7}$$

In particular, (5.6) implies that the Gauss map is quasi-conformal. A map $F : (M_1, g_1) \to (M_2, g_2)$ is said to be *quasi-conformal* if there exists a constant $\Lambda < \infty$ such that for all $x \in M_1$ the ratio of the maximum and minimum eigenvalues of $F^* g_2$ are bounded by Λ. Note that a conformal map is necessarily quasi-conformal (with $\Lambda = 1$).

In the remainder of this section, we will work in local coordinates (x_1, x_2, x_3) with a metric g_{ij} on M. Set e_i equal to the vector field $\frac{\partial}{\partial x_i}$ so that $\langle e_i, e_j \rangle = g_{ij}$. We will use Γ_{ij}^n to denote the Christoffel symbols of the corresponding Riemannian connection.

Suppose that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and consider the graph of the function u

$$\Sigma = \text{Graph}_u = \{(x_1, x_2, u(x_1, x_2)) \mid (x_1, x_2) \in \Omega\} . \tag{5.8}$$

We will first derive the minimal surface equation for Σ in these coordinates. In order to do this, we need to express the mean curvature of Σ in terms of u.

For $i = 1, 2$ we define vector fields E_i and linear maps T_i^n by

$$E_i = e_i + u_{x_i} e_3 \equiv T_i^n\, e_n , \tag{5.9}$$

so that (E_1, E_2) give a basis for the tangent space to Σ. Let $h_{ij} = \langle E_i, E_j \rangle$ denote the induced metric on Σ. It is convenient to define W by

$$W^2 = 1 + g^{ij} u_{x_i} u_{x_j} . \tag{5.10}$$

If N denotes the upward pointing unit normal to Σ, then $\langle N, E_i \rangle = 0$. Therefore, for $i = 1, 2$

$$\langle N, e_i \rangle = -\frac{u_{x_i}}{W} \tag{5.11}$$

and

$$\langle N, e_3 \rangle = \frac{1}{W} . \tag{5.12}$$

The mean curvature is given by

$$h^{ij} \langle N, \nabla_{E_i} E_j \rangle . \tag{5.13}$$

Using (5.9), we compute that

$$h^{ij} = g^{ij} - W^{-2} g^{in} g^{jl} u_{x_n} u_{x_l} \tag{5.14}$$

and

$$\begin{aligned} \nabla_{E_i} E_j &= T_i^n \nabla_{e_n} T_j^l e_l = T_i^n\, T_j^l \nabla_{e_n} e_l + \left(T_i^n \nabla_{e_n} T_j^l \right) e_l \\ &= T_i^n\, T_j^l \Gamma_{nl}^m e_m + E_i(T_j^l)\, e_l = T_i^n\, T_j^l \Gamma_{nl}^m e_m + u_{x_i x_j} e_3 . \end{aligned} \tag{5.15}$$

Combining (5.11), (5.12), and (5.15), we get

$$\langle N, \nabla_{E_i} E_j \rangle = -\frac{1}{W} \sum_{m=1}^{2} \left(u_{x_m} T_i^n\, T_j^l \Gamma_{nl}^m \right) + \frac{1}{W} \left(u_{x_i x_j} + T_i^n\, T_j^l \Gamma_{nl}^3 \right) . \tag{5.16}$$

Multiplying through by W and h^{ij}, we get the minimal surface equation

$$0 = h^{ij} \left(u_{x_i x_j} + T_i^n\, T_j^l \Gamma_{nl}^3 \right) - \sum_{m=1}^{2} \left(u_{x_m} h^{ij} T_i^n\, T_j^l \Gamma_{nl}^m \right) . \tag{5.17}$$

Substituting in the definition of $T_i^n = \delta_i^n + u_{x_i}\,\delta_3^n$, we can write

$$T_i^n\,T_j^l\Gamma_{nl}^m = (\delta_i^n + u_{x_i}\,\delta_3^n)\,(\delta_j^l + u_{x_j}\,\delta_3^l)\,\Gamma_{nl}^m = \Gamma_{ij}^m + u_{x_i}\Gamma_{3j}^m + u_{x_j}\Gamma_{i3}^m + u_{x_i}u_{x_j}\Gamma_{33}^m\,. \tag{5.18}$$

Substituting in (5.18) into (5.17),

$$0 = h^{ij}\,\left(u_{x_ix_j} + \Gamma_{ij}^3 + u_{x_i}\Gamma_{3j}^3 + u_{x_j}\Gamma_{i3}^3 + u_{x_i}u_{x_j}\Gamma_{33}^3\right) - \sum_{m=1}^{2} u_{x_m}h^{ij}\left(\Gamma_{ij}^m + u_{x_i}\Gamma_{3j}^m + u_{x_j}\Gamma_{i3}^m + u_{x_i}u_{x_j}\Gamma_{33}^m\right)\,. \tag{5.19}$$

We next use this to define a function $F : \mathbb{R}^9 \to \mathbb{R}$ given by

(5.20)

$$F(x_1, x_2, u, p_1, p_2, p_{ij}) = h^{ij}\,\left(p_{ij} + \Gamma_{ij}^3 + p_i\Gamma_{3j}^3 + p_j\Gamma_{i3}^3 + p_ip_j\Gamma_{33}^3\right) - \sum_{m=1}^{2} p_m h^{ij}\left(\Gamma_{ij}^m + p_i\Gamma_{3j}^m + p_j\Gamma_{i3}^m + p_ip_j\Gamma_{33}^m\right)\,,$$

where $\Gamma_{ij}^n = \Gamma_{ij}^n(x_1, x_2, u)$ and h^{ij} is the inverse matrix to

(5.21)

$$h_{ij} = g_{ij}(x_1, x_2, u) + p_ig_{j3}(x_1, x_2, u) + p_jg_{3i}(x_1, x_2, u) + p_ip_jg_{33}(x_1, x_2, u)\,.$$

By construction, if u is a solution to the minimal surface equation, then

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_ix_j}) = 0\,. \tag{5.22}$$

If ∇u is bounded, then (5.22) is uniformly elliptic. We conclude that if $|\nabla u|$ is bounded then we get $C^{2,\alpha}$ estimates for u in terms of the maximum of $|u|$.

We will give two further implications of the form of (5.22), first a removable singularities result and then a local description for the intersection of two minimal surfaces.

The following removable singularities result is far from optimal. In fact, results on removable singularities are much stronger for the minimal surface equation than for linear elliptic equations (see, for instance, chapter 10 of [Os2]). However, this result will suffice for our later applications.

Lemma 5.2 *Let u be a C^1 function on $B_1 \subset \mathbb{R}^2$ with*

$$|u| + |\nabla u| \leq C \text{ on } B_1\,. \tag{5.23}$$

If u is a $C^2(B_1 \setminus \{0\})$ solution to the minimal surface equation, then u is a smooth solution on all of B_1.

PROOF: The assumption (5.23) implies that the minimal surface equation is uniformly elliptic with bounded coefficients. Writing this equation in divergence form, the uniform gradient bound (5.23) and the fact that u is a C^2 solution away from 0 imply that u is an H^1 weak solution on all of B_1. Elliptic estimates (theorem 8.8 of [GiTr]) imply that u is in $H^2(B_1)$ so that $|\nabla u| \in H^1(B_1)$. ∎

Suppose now that u and v are both smooth solutions to (5.22). Let $w = v - u$ and for $0 \le s \le 1$ define

$$G(x_1, x_2, s) = F(x_1, x_2, u + sw, (u + sw)_{x_1}, (u + sw)_{x_2}, (u + sw)_{x_i x_j}) . \tag{5.24}$$

By the fundamental theorem of calculus and the chain rule,

$$\begin{aligned} 0 = G(x_1, x_2, 1) - G(x_1, x_2, 0) &= \int_{s=0}^{1} \frac{d}{ds} G(x_1, x_2, s)\, ds \\ &= \int_{s=0}^{1} F_u\, w + F_{p_i}\, w_{x_i} + F_{p_{ij}}\, w_{x_i x_j}\, ds , \end{aligned} \tag{5.25}$$

where each partial derivative of F is evaluated at

$$(x_1, x_2, u + sw, u_{x_1} + sw_{x_1}, u_{x_2} + sw_{x_2}, u_{x_i x_j} + sw_{x_i x_j}) . \tag{5.26}$$

In particular, (5.25) shows that w itself satisfies a partial differential equation where the coefficients are integrals of partials derivatives of F.

This description of the minimal surface equation allows us to apply Bers' theorem, i.e., Theorem 4.27, to decribe the intersections of minimal surfaces in a three-manifold.

Theorem 5.3 (Local Description for the Intersections of Minimal Surfaces) *Suppose that $\Sigma_1^2, \Sigma_2^2 \subset M^3$ are smooth connected immersed minimal surfaces that do not coincide on an open set. Then Σ_1 and Σ_2 intersect transversely except at an isolated set of points $\mathcal{D}$. Given $y \in \mathcal{D}$ there exists an integer $d \ge 2$ and a neighborhood $y \in U$ where the intersection consists of $2d$ embedded arcs meeting at y.*

PROOF: We may assume that $y \in \mathcal{D}$ is a point of nontransverse intersection. Choose coordinates such that $y = (0, 0, 0)$. We may suppose that Σ_1 and Σ_2 are graphs of functions u and v over their common tangent plane at $(0, 0, 0)$. We have that u, v, u_{x_i}, v_{x_i} all vanish at $(0, 0)$. Define w to be $u - v$.

First, we show that w satisfies a uniformly elliptic differential equation with smooth coefficients

$$a_{ij} w_{x_i x_j} + b_i w_{x_i} + cw = 0 . \tag{5.27}$$

Since we are interested in local properties, we may assume that u, v, and their derivatives are uniformly bounded. It follows immediately that the coefficients in (5.25) are smooth and bounded and that a_{ij} is symmetric. In order to check ellipticity, we need to compute

$$a_{ij} = \int_{s=0}^{1} F_{p_{ij}}(x_1, x_2, u + sw, (u + sw)_{x_1}, (u + sw)_{x_2}, (u + sw)_{x_i x_j})\, ds . \tag{5.28}$$

By the mean value theorem (of calculus), for each (x_1, x_2) there exists some $0 \le t \le 1$ such that if $f = u + tw$ then

$$a_{ij}(x_1, x_2) = F_{p_{ij}}(x_1, x_2, f, f_{x_1}, f_{x_2}, f_{x_i x_j}) . \tag{5.29}$$

It is easy to see that $F_{p_{ij}} = h^{ij}$, where h^{ij} is the inverse matrix to

$$h_{ij} = g_{ij} + f_{x_i}\, g_{j3} + f_{x_j} g_{3i} + f_{x_i}\, f_{x_j}\, g_{33} \tag{5.30}$$

and where $g_{nl} = g_{nl}(x_1, x_2, f(x_1, x_2))$. Clearly, h^{ij} is uniformly elliptic if and only h_{ij} is. Finally, since g_{ij} is uniformly elliptic by definition and we may assume that f_{x_i} is arbitrarily small, we may conclude that a_{ij} is uniformly elliptic in some neighborhood $U \subset \mathbb{R}^2$. Since w satisfies (5.27), Theorem 4.27 implies that either w vanishes identically or there exists a linear transformation T such that after rotating by T we can write

$$w(x_1, x_2) = p_d(x_1, x_2) + q(x_1, x_2)\,, \tag{5.31}$$

where p_d is a homogeneous harmonic polynomial of degree $d < \infty$ and

$$|q(x)| + |x|\,|\nabla q(x)| + \cdots + |x|^d\,|\nabla^d q(x)| \leq C|x|^{d+1}\,. \tag{5.32}$$

In particular, this implies that the points of nontransverse intersection are isolated. Finally, it follows immediately from a slight variation of Lemma 4.28 that $u(U) \cap v(U)$ is homeomorphic to $2d$ arcs meeting at x. ∎

Theorem 5.3 and its variation in the case where Σ_i is branched (cf. Theorem 4.33), can be used to extend the results of Chapter 4 to minimal surfaces in a three-manifold M^3.

As an application of Theorem 5.3, we will prove a result of J. Hass on minimal surfaces in three-manifolds with minimal foliations.

Before we explain this result we will need the definition of a lamination (cf., for instance, chapter 8 of W. P. Thurston [Th] or D. Gabai [Ga]).

Definition 5.4 (Lamination) A codimension one *lamination* of M^3 is a collection $\mathcal{L}$ of disjoint smooth connected surfaces (called leaves) such that $\cup_{\Lambda \in \mathcal{L}} \Lambda$ is closed. Moreover, for each $x \in M$ there exists an open neighborhood U of x and a local coordinate chart, (U, Φ), with $\Phi(U) \subset \mathbb{R}^3$ such that in these coordinates the leaves in $\mathcal{L}$ pass through the chart in slices of the form $(\mathbb{R}^2 \times \{t\}) \cap \Phi(U)$.

A *foliation* is a lamination for which the union of the leaves is all of M. The foliation is said to be *oriented* if the leaves are oriented.

Theorem 5.5 (J. Hass [Ha]) *If M^3 has an oriented codimension one foliation with minimal leaves none of which are compact, then M does not contain an immersed minimal $\mathbf{S}^2$.*

PROOF: We will suppose that an immersed minimal sphere $\Sigma \subset M$ exists and deduce a contradiction. Since none of the leaves are compact, Σ is not contained in a leaf. Therefore, Theorem 5.3 implies that if $p \in \Sigma$ with $p \in \Lambda \in \mathcal{L}$ has $T_p\Sigma = T_p\Lambda$, then there is a neighborhood $p \in U$ where they intersect transversely in $U \setminus \{p\}$. In particular, the restriction of the foliation to Σ gives rise to a singular foliation whose vector field vanishes at isolated points (of nontransverse intersection). Applying Theorem 5.3, we see that the index of the zeros of the vector field is negative at each zero. The Poincaré-Hopf formula implies that the sum of these indices is the Euler characteristic of Σ (which is two). This gives a contradiction, and the theorem follows. ∎

Clearly, variations of this argument can be applied more generally to give similar results. Furthermore, certain topological assumptions on M^3 make it possible to prove that a minimal immersed sphere must exist. For instance, J. Sacks and K. Uhlenbeck proved in [SaUh] the existence of an immersed minimal $\mathbf{S}^2$ if $\pi_2(M) \neq 0$. Their ideas also apply to give more general existence results for immersed spheres. Using a completely different approach F. Smith proved in [Sm] that $\mathbf{S}^3$ (with an arbitrary metric) always admits an embedded minimal $\mathbf{S}^2$. In these cases, Theorem 5.5 implies that every minimal foliation of M has compact leaves (see [Ha] for more in this direction).

5.2 Compactness Theorems with A Priori Bounds

In this section, we will prove the following compactness theorem of Choi-Schoen [CiSc]:

Theorem 5.6 (Choi-Schoen [CiSc]) *The space of closed embedded minimal surfaces of genus g in a smooth closed three-manifold M^3 with positive Ricci curvature is compact in the smooth topology.*

The proof of this result uses the positivity of the Ricci curvature to obtain an a priori upper bound on the area of an embedded minimal surface in terms of its genus by combining the following results of Choi-A. N. Wang and P. Yang-Yau:

Theorem 5.7 (Choi-Wang [CiWa]) *If M^n has $\mathrm{Ric}_M \geq \Lambda > 0$ and $\Sigma^{n-1} \subset M$ is a closed embedded minimal hypersurface, then $\lambda_1(\Sigma) \geq \frac{\Lambda}{2}$.*

Theorem 5.8 (Yang-Yau [YgYa]) *Let Σ_g^2 be a closed Riemann surface of genus g; then for any metric on Σ, we have*

$$\lambda_1 \leq \frac{8\pi(1+g)}{\mathrm{Area}\,(\Sigma_g)}\,. \tag{5.33}$$

By combining Theorem 5.7 with the earlier Theorem 5.8, Choi and Wang obtained the following corollary:

Corollary 5.9 (Choi-Wang [CiWa]) *If M^3 has $\mathrm{Ric}_M \geq \Lambda > 0$ and $\Sigma_g^2 \subset M$ is a closed embedded minimal surface with genus g, then*

$$\mathrm{Area}\,(\Sigma_g) \leq \frac{16\,\pi\,(1+g)}{\Lambda}\,. \tag{5.34}$$

Integrating (5.7) and applying the Gauss-Bonnet formula, we get

$$\int_{\Sigma} |A|^2 \leq 2\,k\ \text{Area}\,(\Sigma_g) + 8\,\pi\,(g-1) \leq \frac{32\,\pi\,k\,(1+g)}{\Lambda} + 8\,\pi\,(g-1)\,, \tag{5.35}$$

where the second inequality follows from (5.34). The importance of (5.34) and (5.35) is that the area and total curvature of Σ are bounded uniformly in terms of the genus. This is essential in obtaining a smooth compactness theorem. Note that the area bound alone is enough to give compactness in the space of integral varifolds (see Section 3.1).

The next proposition shows how to use the bounds (5.34) and (5.35) in combination with Theorem 2.3 to obtain a compactness theorem for minimal surfaces (see [CiSc] and cf. M. T. Anderson [An] and B. White [Wh1]).

Proposition 5.10 *Let M^3 be a closed three-manifold and $\Sigma_i \subset M$ a sequence of closed embedded minimal surfaces of genus g with*

$$\text{Area}(\Sigma_i) \leq C_1 \tag{5.36}$$

and

$$\int_{\Sigma_i} |A_{\Sigma_i}|^2 \leq C_2\,. \tag{5.37}$$

There exists a finite set of points $\mathcal{S} \subset M$ and a subsequence $\Sigma_{i'}$ that converges uniformly in the C^ℓ topology (any $\ell < \infty$) on compact subsets of $M \setminus \mathcal{S}$ to a minimal surface $\Sigma \subset M$. The subsequence also converges to Σ in (extrinsic) Hausdorff distance. Σ is smooth in M, has genus at most g, and satisfies (5.36) *and* (5.37).

PROOF: We shall give a proof when $\ell = 2$ since this implies the general case (by standard elliptic theory). Within this proof, $\epsilon = \epsilon(M) > 0$ and $r_0 = r_0(M) > 0$ will be from Theorem 2.3. Let r_1 be such that, by (5.5), for $s < t \leq r_1$ and any minimal surface $\Gamma \subset M$, we have

$$\frac{1}{2} s^{-2}\ \text{Area}(B_s \cap \Gamma) \leq t^{-2}\ \text{Area}(B_t \cap \Sigma)\,. \tag{5.38}$$

In order to find the set $\mathcal{S}$, we define measures ν_i by

$$\nu_i(U) = \int_{U \cap \Sigma_i} |A_i|^2 \leq C_2\,, \tag{5.39}$$

where $U \subset M$ and $A_i = A_{\Sigma_i}$. The general compactness theorem for Radon measures, i.e., Theorem 3.2, implies that there is a subsequence ν_{β_i} which converges weakly to a Radon measure ν with

$$\nu(M) \leq C_2\,. \tag{5.40}$$

For ease of notation, replace ν_i with ν_{β_i}. We define the set $\mathcal{S} = \{x \in M \mid \nu(x) \geq \epsilon\}$. It follows immediately from (5.40) that $\mathcal{S}$ contains at most $C_2\,\epsilon^{-1}$ points.

Given any $y \in M \setminus \mathcal{S}$ we have $\nu(y) < \epsilon$. Since ν is a Radon measure and hence Borel regular, there exists some $0 < 10\,s < \min\{r_0, r_1\}$ (depending on y) such that

$$\nu(B_{10\,s}(y)) < \epsilon\,. \tag{5.41}$$

Since the $\nu_i \to \nu$, (5.41) implies that for i sufficiently large

$$\int_{B_{10\,s}(y)\cap\Sigma_i} |A_i|^2 < \epsilon\,. \tag{5.42}$$

This allows us to apply Theorem 2.3 uniformly to each $B_{10\,s}(y) \cap \Sigma_i$. It follows that for i sufficiently large and $z \in B_{5\,s}(y) \cap \Sigma_i$

$$25\,s^2\,|A_i|^2(z) \leq 1\,. \tag{5.43}$$

By (a slight variation of) Lemma 2.2, (5.43) implies that for each $z \in B_s(y) \cap \Sigma_i$ the connected component of $B_s(z) \cap \Sigma_i$ containing z is a graph over $U^i_z \subset T_z\Sigma_i$ of a function u^z_i with $|\nabla u^z_i| \leq 1$ and $2\,s\,|\nabla^2 u^z_i| \leq 1$. The bounds on $|\nabla u^z_i|$ and $|\nabla^2 u^z_i|$ imply that u^z_i satisfies a uniformly elliptic equation with Lipschitz coefficients (cf. (5.19)). The usual elliptic estimates (see, for instance, corollary 6.3 of [GiTr]) give an $\alpha > 0$ and uniform $C^{2,\alpha}$ estimates for u^z_i on $B_{s/2} \subset T_z\Sigma_i$.

By monotonicity, i.e., (5.38), each connected component of $B_s(y) \cap \Sigma_i$ that intersects $B_{s/2}(y)$ has area at least $\frac{\pi}{8}\,s^2$. Let c_y denote the number of these. By monotonicity, (5.38), and (5.36), we get

$$\frac{\pi}{16}\,r_1^2\,c_y \leq C_1\,. \tag{5.44}$$

In particular, the number of connected components of $B_s(y) \cap \Sigma_i$ which intersect $B_{\frac{s}{2}}(y)$ is bounded independent of both i and y.

Since we have a uniform estimate on the number of components together with uniform $C^{2,\alpha}$ estimates on the graphs, the Arzela-Ascoli theorem gives another subsequence η_i which converges uniformly in $B_s(y)$ in the $C^{2,\frac{\alpha}{2}}$ topology. Since we can cover $M \setminus \mathcal{S}$ by countably many balls like this, a diagonal argument finishes off the convergence to a (possibly immersed) surface Σ which is smooth in $M \setminus \mathcal{S}$. This implies also that Σ satisfies (5.36) and (5.37). Furthermore, the uniform $C^{2,\frac{\alpha}{2}}$ estimates imply that Σ satisfies the same differential equation; that is, Σ is minimal.

We may also suppose that Σ_i converge as integral varifolds to a varifold supported in Σ. By the constancy theorem this must be a multiple of Σ (theorem 41.1 of [Si4]). This convergence implies that monotonicity, i.e., (5.38), applies to the limit Σ. Combining this with the area bound, for any $y \in M$ and $r < r_1$ we have

$$r^{-2}\ \mathrm{Area}(B_r(y) \cap \Sigma) \leq 2\,C_1\,. \tag{5.45}$$

Monotonicity also implies that the Σ_i converge to Σ in Hausdorff distance. To see this, note that if we have $y \in \Sigma_i$ with $\mathrm{dist}(y, \Sigma) > 2\delta$, then monotonicity implies that $B_\delta(y) \cap \Sigma_i$ has area at least $C'\,\delta^2$. Since varifold convergence implies that the area measures converge, we see that the Σ_i must converge to Σ in Hausdorff distance.

Since each Σ_i was embedded and the convergence is smooth, Σ cannot cross itself. However, the local description of Theorem 5.3 implies then that Σ must be embedded.

It remains to show that $\Sigma \cup \mathcal{S}$ is an embedded minimal surface. In other words, we must show that each $x \in \mathcal{S}$ is a removable singularity of Σ. Let A denote the second fundamental form of Σ, so that $|A|^2$ is an L^1 function on Σ (using (5.37)). The monotone convergence theorem implies that

$$\lim_{r\to 0} \int_{B_r(x)\cap\Sigma} |A|^2 = 0\,. \tag{5.46}$$

Hence, given any $0 < \delta < 1$ there exists some $0 < r_x < r_0$ with

$$\int_{B_{2\,r_x}(x)\cap\Sigma} |A|^2 < \delta\,\epsilon\,. \tag{5.47}$$

Applying Theorem 2.3 to Σ itself, if $r < r_x$ and $z \in B_r(x) \setminus B_{r/2}(x)$ then

$$r^2\,|A|^2(z) \leq 4\,\delta\,. \tag{5.48}$$

For the moment, fix $r < r_x$ and $z_1 \in \partial B_{3\,r/4}(x)$. A slight variation of Lemma 2.2 and (2.36) imply that, for δ sufficiently small, the component of $B_{r/4}(z_1) \cap \Sigma$ containing z is a minimal graph with gradient bounded by $C\sqrt{\delta}$ over $T_z\Sigma$. If we repeat this argument for some $z_2 \in \partial B_{3\,r/4}(x) \cap \partial B_{r/8}(z_1)$ in this minimal graph, then we see that the connected component of $B_{r/4}(z_2) \cap \Sigma$ containing z is also a minimal graph with bounded gradient. The area bound (5.45) implies that after iterating this argument around $\partial B_{3\,r/4}(x)$ approximately C_1 times we must close up. Taking δ suffiently small (depending only on C_1), we see that the connected component of $\big(B_r(x) \setminus B_{r/2}(x)\big) \cap \Sigma$ containing z_1 is a graph over a fixed tangent plane with gradient bounded by $C\sqrt{\delta}$ and Hessian bounded by $2\sqrt{\delta}\,r^{-1}$ (see Lemma 2.2).

With $\delta > 0$ small and $r_x > 0$ as above, let Σ_x be a component of $B_{r_x}(x) \cap \Sigma$ with $x \in \bar{\Sigma}_x$. The above discussion shows that Σ_x is a minimal graph of a function u over a fixed tangent plane with $|\nabla u| \leq C\sqrt{\delta}$. We will next see that ∇u has a limit at x. To see this, note that for any $\delta_c > 0$ we may argue as in (5.47) and (5.48) to find some $0 < r_c < r_x$ such that for any $r < r_c$ we have for $z \in \partial B_r(x) \cap \Sigma_x$

$$r^2\,|\,\mathrm{Hess}_u\,| \leq 4r^2\,|A|^2(z) \leq 16\,\delta_c\,. \tag{5.49}$$

Integrating this around $\partial B_r(x)$, and using the fact that $\partial B_r(x) \cap \Sigma_x$ is graphical with bounded gradient, we see that

$$\sup_{z_1,z_2\in\partial B_r(x)\cap\Sigma_x} |\nabla u(z_1) - \nabla u(z_2)| \leq 16\,\pi\,\sqrt{\delta_c}\,. \tag{5.50}$$

It follows immediately that ∇u has a limit at x. In particular, u can be extended to a C^1 solution of the minimal surface equation with uniformly small gradient. We can now apply Lemma 5.2 to conclude that $\Sigma_x \cup \{x\}$ is a smooth minimal surface.

Repeating this for each element of $\mathcal{S}$, we see that $\Sigma \cup \mathcal{S}$ is a smooth minimal surface which is embedded away from $\mathcal{S}$. The local description, i.e., Theorem 5.3, then implies that it is embedded everywhere. This completes the proof. ∎

We are now prepared to prove Theorem 5.6. Since we have area and total curvature bounds in this case, Proposition 5.10 implies that any sequence of closed embedded minimal surfaces of fixed genus will have a subsequence which converges away from finitely many points to a smooth embedded minimal surface. The main remaining point is to show that the convergence holds even across these points. There are now several ways to do this, but we will use the lower bound on λ_1 as in the original proof of [CiSc].

PROOF OF THEOREM 5.6 By Myers' theorem, since $\mathrm{Ric}_M > 0$, M has finite π_1 so that, after passing to a finite cover, we may suppose that M is simply connected. It suffices to show that given any sequence $\Sigma_i \subset M$ of closed embedded minimal surfaces with genus g, there is a subsequence which converges in the $C^{2,\alpha}$ topology for some $\alpha > 0$.

Since $\mathrm{Ric}_M > 0$, (5.34) and (5.35) hold. Hence, we can apply Proposition 5.10. Consequently, there exists a finite set of points $\mathcal{S} \subset M$ and a subsequence $\Sigma_{i'}$ that converges uniformly in the $C^{2,\alpha}$ topology on compact subsets of $M \setminus \mathcal{S}$ to a minimal surface $\Sigma \subset M$. Σ is smooth and embedded in M, has genus at most g, and satisfies (5.34) and (5.35). Since M is simply connected and Σ is embedded, the normal bundle to Σ is trivial.

It remains to show that the convergence is smooth across the points $\mathcal{S}$. We will see that this is equivalent to showing that the convergence is multiplicity one (this equivalence also follows from Allard's regularity theorem). First, we will use Theorem 5.7 to see that the convergence is multiplicity one.

Given any $\epsilon > 0$, there exists $i_0 = i_0(\epsilon)$ such that for any $i > i_0$ we have that

$$\Sigma_i \setminus \bigcup_{x_j \in \mathcal{S}} B_{\epsilon^2}(x_j) = \bigcup_{n=1}^{m} \Sigma_{i,n} \, , \tag{5.51}$$

where each $\Sigma_{i,n}$ is a disjoint minimal graph with bounded gradient over Σ. The facts that the Σ_i are embedded and that the normal bundle to Σ is trivial imply that we may choose a top graph which we label $\Sigma_{i,1}$.

Consequently, if the convergence is not multiplicity one, then for large i the minimal surface Σ_i consists of large graphical pieces connected in small sets. Using this description, we will construct a test function which is constant except near these small sets. This will be shown to violate the lower bound on λ_1.

Define a Lipschitz function ϕ on Σ_i by

$$\phi(x) = 1 \text{ for } x \in \Sigma_{i,1} \setminus \bigcup_{x_j \in \mathcal{S}} B_{\epsilon}(x_j) \, ,$$

$$\phi(x) = \frac{\log \mathrm{dist}_M(x, x_j) - 2 \log \epsilon}{\log \epsilon - 2 \log \epsilon} \text{ for } x \in \Sigma_{i,1} \cap B_{\epsilon}(x_j) \setminus B_{\epsilon^2}(x_j) \, ,$$

$$\phi(x) = 0 \text{ for } x \in \Sigma_{i,1} \cap \left(\bigcup_{x_j \in \mathcal{S}} B_{\epsilon^2}(x_j) \right) ,$$

$$\phi(x) = - \frac{\log \operatorname{dist}_M(x, x_j) - 2 \log \epsilon}{\log \epsilon - 2 \log \epsilon} \text{ for } n > 1 \, , x \in \Sigma_{i,n} \cap B_\epsilon(x_j) \setminus B_{\epsilon^2}(x_j) \, ,$$

$$\phi(x) = -1 \text{ for } n > 1 \text{ and } x \in \Sigma_{i,n} \setminus \bigcup_{x_j \in \mathcal{S}} B_\epsilon(x_j) \, .$$

Set $c = \int_{\Sigma_i} \phi$ so that $\psi = \phi - c$ has integral zero. Clearly, we have that

$$|\psi| \geq 1 + |c| \geq 1 \tag{5.52}$$

on at least one of the sheets, and hence

$$\lim_{\epsilon \to 0} \int_{\Sigma_i} \psi^2 \geq \operatorname{Area}(\Sigma) \, . \tag{5.53}$$

On the other hand, $\nabla \psi = \nabla \phi$ and hence the coarea formula (i.e., (1.40)) gives

$$\begin{aligned} \int_{\Sigma_i} |\nabla \psi|^2 &\leq \frac{1}{(\log \epsilon)^2} \sum_{x_j \in \mathcal{S}} \int_{B_\epsilon(x_j) \setminus B_{\epsilon^2}(x_j)} \frac{|\nabla r_j|^2}{r_j^2} \\ &= \frac{1}{(\log \epsilon)^2} \sum_{x_j \in \mathcal{S}} \int_{t=\epsilon^2}^{\epsilon} \int_{\{r_j = t\} \cap \Sigma_i} \frac{|\nabla r_j|}{t^2} \, , \end{aligned} \tag{5.54}$$

where $r_j(x) = \operatorname{dist}(x, x_j)$. We can estimate this using Stokes' theorem and monotonicity to get

$$\begin{aligned} \int_{\{r_j = t\} \cap \Sigma_i} \frac{|\nabla r_j|}{t^2} &= \frac{1}{2} \, t^{-3} \int_{B_t(x_j) \cap \Sigma_i} \Delta r_j^2 \\ &\leq 3 \, t^{-2} \, \operatorname{Area}(B_t(x_j) \cap \Sigma_i) \leq 3 \, C_1 \, t^{-1} \, . \end{aligned} \tag{5.55}$$

Substituting (5.55) into (5.54) and integrating,

$$\int_{\Sigma_i} |\nabla \psi|^2 \leq 3 \, C_1 \, |\mathcal{S}| \, \frac{1}{(\log \epsilon)^2} \int_{\epsilon^2}^{\epsilon} \frac{1}{t} \, dt = 3 \, C_1 \, |\mathcal{S}| \, \frac{1}{\log \epsilon} \, . \tag{5.56}$$

Therefore, taking ϵ sufficiently small, the test function ψ violates Theorem 5.7. This contradiction shows that $m = 1$, that is, the convergence is multiplicity one.

Using this, we will get uniform curvature estimates for the Σ_i in a neighborhood of each x_j. As before, combining this with the Arzela-Ascoli theorem will imply smooth convergence even across $\mathcal{S}$.

Given $x_j \in \mathcal{S}$ and $0 < \epsilon$, since Σ is smooth we may choose an $0 < s$ such that $B_{2s}(x_j) \cap \Sigma$ is a minimal graph with gradient bounded by ϵ. For i sufficiently large we may suppose that $(B_{2s}(x_j) \setminus B_s(x_j)) \cap \Sigma_i$ is the graph over Σ of a function u_i with $s^{-1} \, |u| + |\nabla u| \leq 2\epsilon$. Since the $\Sigma_i \to \Sigma$ in Hausdorff distance, for i sufficiently large we also have Σ_i contained in an $\frac{s}{4}$ tubular neighborhood. Note that the fact that the convergence was multiplicity one was used to conclude that this was a single graph. Since Σ_i is contained in the tubular neighborhood, the boundary of $B_{2s}(x_j) \cap \Sigma_i$ is $\partial B_{2s}(x_j) \cap \Sigma_i$. Using this description as a graph with small gradient and applying Stokes' theorem to $\Delta_{\Sigma_i} r_j^2$, we get

$$\frac{\text{Area}(B_{2s}(x_j) \cap \Sigma_i)}{4\,\pi\,s^2} \leq 1 + C_3\,\epsilon\,. \tag{5.57}$$

By monotonicity, (5.57) implies that for any $z \in B_s(x_j) \cap \Sigma_i$ we have

$$\frac{\text{Area}(B_s(z) \cap \Sigma_i)}{\pi\,s^2} \leq 1 + C_4\,\epsilon\,. \tag{5.58}$$

Finally, the smooth version of Allard's regularity theorem, i.e., Theorem 2.10, yields uniform curvature estimates for $\epsilon > 0$ sufficiently small, and the proof is complete. ∎

5.3 Convergence of Embedded Minimal Surfaces

In this section, we will discuss (without proofs) some very recent results (see [CM3], [CM8], [CM9], and [CM6]) on convergence and compactness of embedded minimal surfaces in three-manifolds. The emphasis here will be on the case where there is no a priori area or total curvature bound.

Part of the motivation for studying convergence and compactness of embedded minimal surfaces in three-manifolds comes from the following question whose positive answer would have many important applications to the topology of three-manifolds:

Question 5.11 (Pitts-Rubinstein; see problem 29 in [Ya]) *Let M^3 be a closed simply connected Riemannian three-manifold. Does there exist a bound for the Morse index of all closed embedded minimal surfaces with fixed genus in M? Pitts and Rubinstein say that if this is the case for a sufficiently large class of metrics on all closed simply connected three-manifolds, then the famous space-form problem may be solved affirmatively.*

The *spherical space-form problem*, which dates back to the nineteenth century, asks to show that any free action of a finite group on the 3-sphere is topologically conjugate to an orthogonal action.

The results of [CiSc] (see Proposition 5.10, in particular) can be applied to obtain bounds on the Morse index of minimal surfaces of bounded area and fixed genus in a closed three-manifold; see the appendix of [CM6]. It follows that the key to solving the question of Pitts-Rubinstein is to understand what happens when no such area bound exists. Thus, the approach of [CM3], [CM8], [CM9] and [CM6] to answering the question of Pitts-Rubinstein is to study sequences of closed embedded minimal surfaces (possibly without a uniform area bound) in a fixed closed three-manifold. In particular, given such a sequence, the approach taken there is to extract a convergent subsequence and to study the regularity of the limit and the convergence.

We will briefly discuss here both the global question of convergence of closed minimal surfaces in three-manifolds, cf. Example 5.12, and its local analog, cf. Example 5.13.

Example 5.12 Let $\Sigma_i \subset T^3$ be a sequence of totally geodesic embedded tori with $\text{Area } \Sigma_i \to \infty$.

Example 5.13 Let $\Sigma_i \subset B_1 \subset \mathbb{R}^3$ be a sequence of rescaled helicoids (see Example 1.3).

In each example, the sequence is not compact in the classical sense. However, by considering each minimal surface Σ_i as a Radon measure μ_i on $B_1(0) \subset \mathbb{R}^n$ (cf. (3.4)) and then renormalizing to get probability measures (i.e., replacing μ_i with $\mu_i/\mu_i(B_1)$), we obtain a sequence of Radon measures with uniformly bounded mass which has a weak limit (see (3.3) for the definition of weak convergence of Radon measures). In fact, the associated renormalized varifolds (see Section 3.1) may be seen to converge (although the limit is not a rectifiable varifold). In general, if we have a sequence of surfaces, then, after renormalizing the associated measures, we can extract a weakly convergent subsequence from Theorem 3.2.

Observe also that the second example shows that no local curvature estimate need hold in general for embedded minimal surfaces (even without neck pinchings). This is in contrast to the case of embedded minimal surfaces with uniformly bounded area, where a neck pinching is the only way that a curvature estimate can fail to hold.

The main result of [CM8] about convergence is stated for sequences of surfaces called uniformly locally simply connected. There are similar results for sequences of surfaces of fixed genus, but the assumption that the surfaces are uniformly locally simply connected simplifies the statements. These surfaces are defined as follows:

Definition 5.14 (Uniformly Locally Simply Connected) Let M^n be an n-dimensional manifold with curvature bounded above by k and injectivity radius bounded below by $i_0 > 0$. We say that a sequence of surfaces $\Sigma_i^2 \subset M$ is *uniformly locally simply connected* (or ULSC) if for some R with $0 < 2R < \min\{\frac{i_0}{4}, \frac{\pi}{4\sqrt{k}}\}$ there exists a covering of M by balls, $\{B_R(y_j)\}_j$, such that for each i, j, each connected component of $B_{2R}(y_j) \cap \Sigma_i$ is simply connected.

A simple application of the maximum principle shows that a simply connected minimal surface in $\mathbb{R}^n$ is automatically uniformly locally simply connected. This will be used in the next section (see also [CM7]) where we will apply the results described here to study complete, properly embedded, minimal surfaces in $\mathbb{R}^3$ with finite topology.

The main result of [CM8] about convergence is modelled on Example 5.13. This result is the following (see the previous section for the definition of a lamination):

Theorem 5.15 (Colding-Minicozzi, Theorem 0.10 in [CM8]) *Let M^3 be a closed three-dimensional manifold and let Σ_i^2 be a sequence of ULSC closed embedded minimal surfaces in M. After going to a subsequence, there is a limit* *measure μ, a closed subset, $\mathcal{S} \subset M$, with*

$$\mu_{-2}(\mathcal{S}) \leq m < \infty \,, \tag{5.59}$$

such that $\Sigma_i \setminus \mathcal{S}$ converges in the C^α topology for any $\alpha < 1$ to a lamination in $M \setminus \mathcal{S}$ with minimal leaves. Here $m = m(i_0, k, \mathrm{diam}, R)$.

In Theorem 5.15, as well as in Theorem 5.25 below, the R is the constant occurring in the definition of uniformly locally simply connected, Definition 5.14.

There is a version of this theorem (see [CM10]) which hold for sequences of minimal surfaces of fixed genus (instead of ULSC). For the sake of simplicity, we will not state this more general result here since the precise statement is more complicated and can anyway essentially be reduced to the above. We will now briefly describe this reduction.

First, the uniformity of $R > 0$ in the definition of ULSC is not essential. In general, we may allow $R > 0$ to depend on the point x (but not of course on i). Let Σ_i be a sequence of closed embedded minimal surfaces of genus g. Suppose now that x is a point where for every $R > 0$ we have infinitely many i such that $B_R(x) \cap \Sigma_i$ fails to be simply connected.

There are two quite different ways in which a sequence of surfaces of fixed genus can can fail to be ULSC at some point. Namely, the corresponding sequence of (locally) homotopically nontrivial curves which is shrinking to the point can either divide the surface or not. In the first case, the surface is locally a planar domain. In the second case, the surface intersected with $B_R(x)$ is said to have positive genus. We will ignore the second case since (after passing to a subsequence) there are only finitely many points where this occurs. We may therefore conclude that if a sequence fails to be ULSC at x, then after going to a subsequence there are curves $\gamma_i \subset B_{R_i}(x) \cap \Sigma_i$ with $R_i \to 0$ so that for some $R > 0$,

$$\begin{aligned} &\gamma_i \text{ is null homotopic in } \Sigma_i \,, \\ &\gamma_i \text{ is not contractible in } B_R(x) \cap \Sigma_i, \text{ and} \\ &\gamma_i \text{ divides } B_R(x) \cap \Sigma_i \,. \end{aligned} \tag{5.60}$$

It is then possible to show that for each i there is a stable surface in the complement of Σ_i which intersects $B_{R_i}(x)$ and that these converge to a smooth stable surface through x. Moreover, with additional work, it follows that this stable surface lies in the Hausdorff limit of the surfaces Σ_i. In this way, we see that the main case is when the surfaces are uniformly locally simply connected, where $R = R(x) > 0$.

By rescaling the helicoid (see Example 5.13) we get a sequence of minimal surfaces which converge smoothly to a lamination by flat planes off the vertical axis. Since the renormalized limit measure in that case is a multiple of Lebesgue measure, the singular set has positive codimension two measure. Hence, we see that the estimate on the singular set in Theorem 5.15 is optimal. In fact, one consequence of Theorem 5.15 and its proof is that the behavior of the helicoid is typical.

We will now describe some of the ideas and results that go into the proof of the above convergence result, many of which have independent interest. In the next section, we will discuss an application of some of these results to complete minimal surfaces in $\mathbb{R}^3$.

The first result that we will mention gives a bound on the area for a minimal surface whose average curvature is small in an appropriate sense. Namely, in [CM8] the following is shown:

Proposition 5.16 (Colding-Minicozzi, Corollary 2.37 in [CM8]) *There exists a constant $\rho = \rho(n, i_0, k) > 0$ and a constant $\epsilon_1 > 0$ such that the following holds: Let $x \in M^n$ and $r_0 < \rho$. If $x \in \Sigma^2 \subset M$ is a compact minimal immersed surface with boundary contained in $\partial B_{7\,r_0}(x)$, Σ' is a connected component of $B_{r_0} \cap \Sigma$, and for some $\epsilon > 0$*

$$r_0^2 \int_{B_{7\,r_0} \cap \Sigma} |A|^2 < \epsilon \, \text{Area}(\Sigma') \, , \tag{5.61}$$

then

$$\left(1 - 2r_0^2 k - 36 r_0 \sqrt{k} - 40\,\pi \left(\frac{1}{\epsilon_1} + \frac{1}{4\pi}\right) \epsilon\right) \text{Area}(\Sigma') \leq \pi \, r_0^2 \, . \tag{5.62}$$

This proposition has a number of consequences. Here we will only state one (see also [CM8] for more results in the same spirit).

Combining this proposition with Theorem 2.3 gives the following curvature estimate:

Corollary 5.17 (Colding-Minicozzi, Corollary 2.49 in [CM8]) *There exist $0 < \epsilon_0 = \epsilon_0(n, i_0, k) < 1$, $\rho = \rho(n, i_0, k) > 0$, and $C = C(n, i_0, k) < \infty$ such that if $\Sigma^2 \subset M^n$ is a compact minimal immersed surface with boundary contained in $\partial B_{7\,r_0} = \partial B_{7\,r_0}(x)$, $r_0 \leq \rho$, and for some $\epsilon \leq \epsilon_0$ and $0 < s \leq r_0$*

$$\int_{B_{7\,r_0} \cap \Sigma} |A|^2 < \epsilon \, s^{-2} \; \text{Area}(B_s \cap \Sigma) \, , \tag{5.63}$$

then for some connected component of $B_{r_0} \cap \Sigma$, Σ', which intersects B_s and any $0 < \sigma \leq r_0$

$$\sup_{B_{r_0 - \sigma} \cap \Sigma'} |A|^2 \leq C \, \epsilon \, \sigma^{-2} \, . \tag{5.64}$$

Corollary 5.17 should be compared with the regularity result for small *total* curvature (Theorem 2.3).

A key difficulty in the proof of Theorem 5.15 is the very complicated local behavior of minimal surfaces without area bounds. For instance, as explained above, there is no uniform bound on the area, the curvature, or the number of connected components in small balls. Furthermore, this local behavior could be very different at different points and on different scales.

This last point is an important one; often in geometric analysis, monotonicity of some (scale-invariant) quantity is used to show that there is a uniform structure on all scales (away from a small subset). For minimal surfaces, both the density (by Proposition 1.8) and the total curvature are (scale-invariant) monotone quantities, and thus if some ball has small density or small total curvature, then so do all sufficiently small subballs. This is an important feature in the curvature estimates for small density and small total curvature (Theorems 2.10 and 2.3). In fact, examples show that the "small average curvature" condition of Corollary 5.17 does not necessarily hold on smaller subballs; that is, there is no monotonicity in this setting.

We have seen that sufficiently small bounds on the density and total curvature lead to Bernstein-type theorems and curvature estimates. In fact, various other scale-invariant hypotheses also suffice. To illustrate this, we will now mention several Bernstein-type theorems of independent interest.

The first is the following:

Proposition 5.18 (Colding-Minicozzi, Corollary 5.21 in [CM8]) *Let $\Sigma^2 \subset \mathbb{R}^n$ be a complete properly embedded minimal surface with genus g and polynomial volume growth. If for some $r_0 \geq 0$ the intersection of Σ with every ball centered on Σ and of radius at least r_0 has at most ℓ connected components, then Σ has finite total curvature and at most ℓ ends.*

The second is an immediate consequence of the following curvature estimate:

Proposition 5.19 (Colding-Minicozzi, Corollary 5.24 in [CM8]) *There exist two constants $C = C(g,n) < \infty$ and $\rho = \rho(n, i_0, k) > 0$ such that if $\Sigma^2 \subset B_{r_0} = B_{r_0}(x) \subset M^n$ is a compact embedded minimal surface of genus g, with $\partial\Sigma \subset \partial B_{r_0}$, $r_0 \leq \rho$, and the intersection of Σ with all balls centered on Σ and contained in B_{r_0} is connected, then for all $0 < \sigma \leq r_0$,*

$$\sup_{B_{r_0-\sigma}\cap\Sigma} |A|^2 \leq C\,\sigma^{-2}\,. \tag{5.65}$$

Corollary 5.20 (Colding-Minicozzi, Corollary 5.33 in [CM8]) *Let $\Sigma^2 \subset \mathbb{R}^n$ be a complete properly embedded minimal surface with genus at most g. If its intersection with every ball centered on Σ is connected, then Σ is a plane.*

The two previous propositions and Corollary 5.20 give estimates for embedded minimal surfaces with finite topology whose intersections with all balls are connected (or, more generally, have a bounded number of components). On the other hand, it is shown in [CM8] that if a minimal surface is "disconnected on all scales," then it must be flat. To make this precise, inspired by the classical Reifenberg condition (see [Re], and cf. also with [Mo2], [To] and [ChC]) the following definition is made in [CM8]:

Definition 5.21 (One-Sided Reifenberg Condition) A subset, Γ, of M^n is said to satisfy the (δ, r_0)-*one-sided Reifenberg condition* at $x \in \Gamma$ if for every $0 < \sigma \leq r_0$ and every $y \in B_{r_0-\sigma}(x)\cap\Gamma$, there corresponds a connected hypersurface, $L^{n-1}_{y,\sigma}$, with $\partial L_{y,\sigma} \subset \partial B_\sigma(y)$,

$$B_{\delta\sigma}(y) \cap L_{y,\sigma} \neq \emptyset \, , \tag{5.66}$$

$$\sup_{B_\sigma(y)\cap L} |A_L|^2 \leq \delta^2 \, \sigma^{-2} \, , \tag{5.67}$$

and such that the connected component of $B_\sigma(y) \cap \overline{\Gamma}$ through y lies on one side of $L_{y,\sigma}$.

For embedded minimal hypersurfaces which satisfy the one-sided Reifenberg condition, the following curvature estimate is proven in [CM8]:

Theorem 5.22 (Colding-Minicozzi, Theorem 4.11 in [CM8]) *There exist $\epsilon = \epsilon(n, i_0, k) > 0$ and $\rho = \rho(n, i_0, k) > 0$ such that the following holds: If $r_0 \leq \rho$, $\Sigma^{n-1} \subset B_{r_0} = B_{r_0}(x) \subset M^n$ is a compact embedded minimal hypersurface with $\partial\Sigma \subset \partial B_{r_0}$ and Σ satisfies the (ϵ, r_0)-one-sided Reifenberg condition at x, then for all $0 < \sigma \leq r_0$*

$$\sup_{B_{r_0-\sigma}\cap\Sigma} |A|^2 \leq \sigma^{-2} \, . \tag{5.68}$$

An immediate consequence of this curvature estimate is the following Bernstein-type theorem:

Corollary 5.23 (Colding-Minicozzi, Corollary 4.18 in [CM8]) *There exists $\epsilon = \epsilon(n) > 0$ such that if $\Sigma^{n-1} \subset \mathbb{R}^n$ is a connected properly embedded minimal hypersurface which satisfies the (ϵ, ∞)-one-sided Reifenberg condition, then Σ is a flat hyperplane.*

For the proof of the main theorem about convergence (Theorem 5.15), we will unfortunately never be in a situation where the minimal surfaces come close on all sufficiently small scales to a surface with small curvature and lie on one side of it. Rather, we will have for an open subset, whose complement has codimension two, that every point is the center of *some* ball in which every minimal surface in a subsequence comes close to a surface with small curvature and lies on one side of it. This then leads one to consider an effective version of the above results. Namely, in [CM8] a curvature estimate is proven for a simply connected properly embedded minimal surface, Σ^2, in a three-manifold M^3 which in a ball lies on one side of, but comes close to, a surface L^2 with small curvature. A minimal surface which has this property is said to satisfy the *effective one-sided Reifenberg condition*. Note that this property is not required to hold on all sufficiently small scales. Unlike in Theorem 5.22, the simple connectivity of Σ^2 is essential, as can be seen from the catenoid.

Theorem 5.24 (Main Technical Curvature Estimate, Theorem 6.4 in [CM8]) *There exist $\rho = \rho(i_0, k)$, $\delta = \delta(i_0, k) > 0$, $\epsilon > 0$, and a constant $C \geq 1$ such that the following holds. Let $0 < r_0 < \rho$ and Σ^2, $L^2 \subset B_{2r_0} \subset M^3$ be compact connected embedded surfaces in a three-manifold M with $\partial\Sigma$, $\partial L \subset \partial B_{2\,r_0}$.*

Suppose that $B_{\epsilon r_0} \cap \Sigma \neq \emptyset$, $B_{\epsilon r_0} \cap L \neq \emptyset$, *and*

$$\sup_{B_{2r_0} \cap L} |A_L|^2 \leq \delta\, r_0^{-2} \,. \tag{5.69}$$

If Σ *is minimal and simply connected and lies on one side of* L*, then for any connected component* Σ' *of* $B_{\frac{r_0}{2}} \cap \Sigma$ *with* $B_{\epsilon r_0} \cap \Sigma' \neq \emptyset$ *we have*

$$\sup_{B_{\frac{r_0}{4}} \cap \Sigma'} |A_\Sigma|^2 \leq C\, r_0^{-2} \,. \tag{5.70}$$

It is important that Theorem 5.24 does not make any a priori assumptions on the minimal surface other than that it is a properly embedded disk. By way of contrast, the classical curvature estimates for minimal surfaces either assume a priori geometric bounds (on area or total curvature bound), or stability (with estimates for graphs being a special case).

In [CM8], Proposition 5.16 and its corollary are used to show that any sequence of ULSC closed embedded minimal surfaces in a three-manifold has a subsequence which satisfies a uniform effective one-sided Reifenberg condition away from a closed subset of finite codimensional two measure. Combining this with Theorem 5.24, one gets uniform curvature estimates for all minimal surfaces in the subsequence away from such a subset. It is then possible to conclude that a subsequence converges smoothly to a lamination away from a closed subset of finite codimensional two measure.

Theorem 5.25 (Curvature Estimates Outside a Closed Subset of Codimension Two, Theorem 8.39 in [CM8]) *Let* M^3 *be a closed three-dimensional manifold and let* Σ_i^2 *be a sequence of ULSC closed embedded minimal surfaces in* M*. Let* μ_i *denote the renormalized measures. After going to a subsequence, there is a limit measure* μ*, a closed subset,* $\mathcal{S} \subset M$*, with*

$$\mu_{-2}(\mathcal{S}) \leq m < \infty \,, \tag{5.71}$$

and for all $x \in M \setminus \mathcal{S}$*, there exists a* $r_0 > 0$ *(depending on* x*) such that for all* i *sufficiently large*

$$\sup_{B_{4r_0} \cap \Sigma_i} |A_i|^2 \leq r_0^{-2} \,. \tag{5.72}$$

Here $m = m(i_0, k, \mathrm{diam}, R)$ *and* $B_{4r_0} = B_{4r_0}(x)$*.*

The main convergence result, Theorem 5.15, is then shown using Theorem 5.25.

5.4 Complete Embedded Minimal Surfaces in $\mathbb{R}^3$

The results of the previous section also have applications for complete minimal surfaces in $\mathbb{R}^3$. This section concerns one such application, namely to the ends of a properly embedded complete minimal surface $\Sigma^2 \subset \mathbb{R}^3$ with finite topology.

For such surfaces each end has a representative E which is a properly embedded minimal annulus. If E has finite total curvature, it is asymptotic to either a plane or half of a catenoid. On the other hand, the helicoid provides the only known example of an end with finite topology and infinite total curvature. Clearly, no representative for the end of the helicoid can be disjoint from an end of a plane or catenoid.

The main result of [CM7] is that any complete properly embedded minimal annulus which lies above a sufficiently narrow downward sloping cone must have finite total curvature. This is closely related to a result of P. Collin [Co] described below.

In [HoMe], D. Hoffman and W. Meeks proved that at most two ends of Σ can have infinite total curvature. Further, they conjectured that if Σ as above has at least two ends, then Σ must have finite total curvature (this is the "finite total curvature conjecture"; cf. [Me]).

If Σ has at least two ends, then there is either an end of a plane or a catenoid disjoint from Σ by lemma 5 of [HoMe]. Therefore, to prove the finite total curvature conjecture, it suffices to show that a properly embedded minimal annular end E which lies above the bottom half of a catenoid has finite total curvature.

In this direction, W. Meeks and H. Rosenberg [MeR] showed that if Σ has at least two ends, then Σ is *conformally* equivalent to a compact Riemann surface with finitely many points removed. Using this, they showed that an annular end E arising in the finite total curvature conjecture lies in a half-space. In fact, they showed that E is either asymptotically planar (with finite total curvature) or satisfies the hypotheses of the "generalized Nitsche conjecture."

Conjecture 5.26 (Generalized Nitsche Conjecture [MeR]; Collin's Theorem, [Co]) *For $t \geq 0$ let $P_t = \{x_3 = t\} \subset \mathbb{R}^3$. Suppose that $E \subset \{x_3 \geq 0\}$ is a properly embedded minimal annulus with $\partial E \subset P_0$ and that E intersects every P_t for $t > 0$ in a simple closed curve; then E has finite total curvature.*

P. Collin proved this conjecture in [Co], thereby showing, using [MeR], that, for properly embedded complete minimal surfaces with at least two ends, finite topology is equivalent to finite total curvature. An example of a properly immersed minimal cylinder in $\mathbb{R}^3$ with infinite total curvature, constructed by of H. Rosenberg and E. Toubiana [RT], shows that embeddedness is a necessary hypothesis in both the finite total curvature and generalized Nitsche conjectures. Well-known examples show that properness is also necessary.

Let x_1, x_2, x_3 be the standard coordinates on $\mathbb{R}^3$. Given any $\epsilon \in \mathbb{R}$ we let $\mathcal{C}_\epsilon \subset \mathbb{R}^3$ denote the conical region

$$\{x_3 > \epsilon \sqrt{x_1^2 + x_2^2}\} . \tag{5.73}$$

With this definition, $\mathcal{C}_0$ is a halfspace and $\mathcal{C}_\epsilon$ is convex if and only if $\epsilon \geq 0$.

Theorem 5.27 (Colding-Minicozzi [CM7]) *There exists $\epsilon > 0$ such that any complete properly embedded minimal annular end $E \subset \mathcal{C}_{-\epsilon}$ has finite total curvature.*

The generalized Nitsche conjecture follows directly from Theorem 5.27 since $\{x_3 \geq 0\} = \mathcal{C}_0 \subset \mathcal{C}_{-\epsilon}$. In fact, since the height of the catenoid only grows logarithmically, the finite total curvature conjecture also follows directly from Theorem 5.27.

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